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EUCLID: BOOKS V., VI., XI.

BY

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PREFACE.

THIS book completes the edition of Euclid's Elements which was begun by the publication of Books I.–IV. in 1897. That part was so favourably received that the author has adopted a similar method of treating the subject in this instalment of the work. At the end of Book VI. an Abstract of the Suggestions made by a Committee of the Mathematical Association in 1902 has been given; and a Summary of Results is given at the end of every one of Euclid's Books. It will, therefore, be easy to use this edition of Euclid side by side with a book on Geometrical Drawing in the manner recommended by the Committee of the Cambridge University Syndicate, and in accordance with approved methods of teaching Geometry. In several cases Alternative Proofs of Propositions have been given, and other proofs will be found among the Exercises, so that the student may adopt whichever proof he thinks best.

The author takes this opportunity of acknowledging, with thanks, the many appreciative letters and criticisms of the edition of Books I.–IV. which he has received.

RUPERT DEAKIN.

KING EDWARD'S GRAMMAR SCHOOL,
STOURBRIDGE,
January, 1903.

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EUCLID'S ELEMENTS.

BOOKS V. AND VI.

EUCLID'S ELEMENTS.

BOOK V.

ON RATIO AND PROPORTION.

DEFINITIONS.

1. A less magnitude is said to be a *submultiple* of a greater magnitude when the less is contained a certain number of times exactly in the greater.

2. A greater magnitude is said to be a *multiple* of a less magnitude when the greater contains the less a certain number of times exactly.

3. *Ratio* is the relation of two quantities of the same kind to each other in respect of magnitude, this relation being determined by division.

The ratio of A to B is written $A : B$, and this is read "A *is to* B."

A and B are called the *terms* of the ratio, A being the *antecedent* and B the *consequent* of the ratio.

A *ratio of equality* is a ratio in which the antecedent is equal to the consequent. A ratio of equality is said to be equal to unity.

A *ratio of greater inequality* is a ratio in which the antecedent is greater than the consequent. A ratio of greater inequality is said to be greater than unity.

A *ratio of less inequality* is a ratio in which the antecedent is less than the consequent. A ratio of less inequality is said to be less than unity.

Notes on the Definitions.

1. Euclid uses the word *part* instead of submultiple. This is objectionable because the word *part* is also used to denote any portion less than all the magnitude, and so a part need not be an aliquot part of the magnitude. For example, 3 inches is a part of 10 inches, but 3 inches is not an aliquot part or submultiple of 10 inches.

Some writers use the expression *measure* or *aliquot part* instead of the word "submultiple."

Equimultiples of magnitudes are multiples which contain the magnitudes the same number of times. For example, a foot and a shilling are equimultiples of an inch and a penny respectively. Generally, if $A = mC$ and $B = mD$, where m is any whole number, then A and B are equimultiples of C and D.

3. The student will do well to read the chapters on Ratio and Proportion in Algebra before going further in Euclid, Book V. (See *Tutorial Algebra*, Part I., Chapters XXV., XXVI.)

The relation of one magnitude to another of the same kind may be determined in two ways, viz.: (1) by subtracting the less magnitude from the greater—this does *not* determine their ratio; and (2) by determining how many times some aliquot part of one magnitude is contained in the other magnitude—this determines their ratio, and the process is of the nature of division.

4. Two multiples are said *to have the same ratio* as two other magnitudes when—

if any equimultiples whatever be taken of the 1st and 3rd, and any equimultiples whatever be taken of the 2nd and 4th,

$$\left\{\begin{matrix}\text{the multiple}\\ \text{of the 1st}\\ \text{is}\end{matrix}\right\}\left\{\begin{matrix}\text{greater than,}\\ \text{equal to,}\\ \text{or less than}\end{matrix}\right\}\left\{\begin{matrix}\text{the multiple}\\ \text{of the 2nd}\end{matrix}\right\}$$

$$\text{according as}\left\{\begin{matrix}\text{the multiple}\\ \text{of the 3rd}\\ \text{is}\end{matrix}\right\}\left\{\begin{matrix}\text{greater than,}\\ \text{equal to,}\\ \text{or less than}\end{matrix}\right\}\left\{\begin{matrix}\text{the multiple}\\ \text{of the 4th}\end{matrix}\right\}.$$

5. Four magnitudes are said to be *proportionals*, or *in proportion*, when the ratio of the first two magnitudes is the same as the ratio of the last two.

Thus, if $A : B$ is the same ratio as $C : D$, then A, B, C, D are proportionals. This is usually expressed thus—

$$A : B :: C : D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

which is read "A is to B as C is to D";

or thus— $$A : B = C : D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

which may be read in the same way as (1), or thus—"the ratio of A to B equals the ratio of C to D."

In Geometry it is perhaps better to use (1), because the sign = implies that the ratios can be expressed exactly by numbers, which is impossible if the magnitudes are incommensurable.

6. When four magnitudes are proportionals, the two antecedents are said to be *homologous* or *corresponding terms;* so also are the consequents.

Thus, if $A : B :: C : D$, A and C, the antecedents, are homologous or corresponding terms; so also are the consequents B and D.

7. Magnitudes are said to be in *continued proportion* if, all the magnitudes being of the same kind, the first is to the second as the second is to the third, and as the third is to the fourth, and so on.

Thus A, B, C, D, E, F, &c., are in continued proportion if they all denote magnitudes of the same kind and if $A : B :: B : C :: C : D$, &c.

If we can find some aliquot part of one magnitude which is contained an exact number of times in the other magnitude—in other words, if the two magnitudes have a common measure—the two magnitudes are said to be *commensurable*. If, however, the two magnitudes have no common measure, they are said to be *incommensurable*. For example, the ratio of a diagonal to the side of a square and the ratio of the circumference of a circle to a diameter of the same circle are both incommensurable. In such cases, the ratio in question cannot be expressed exactly in numbers; the ratio may, however, be expressed to any assigned degree of accuracy by taking numbers sufficiently large.

Again, it is important to notice that a ratio only exists between quantities *of the same kind*. We cannot compare two things of different kinds—for example, an apple and a pear. We may, however, compare the size of the apple with the size of the pear; then, however, we are comparing sizes, and not the apple with the pear. We abstract the sizes, and consider them apart from the apple and the pear. Now, in Plane Geometry, magnitudes are of three kinds, viz.: Lines, Angles, and Areas. The ratios therefore may refer to two lines, two angles, or two areas. There is no ratio between a line and an angle, or between a line and an area, because the magnitudes are not of the same kind.

4. This definition is Euclid's fifth definition, and it applies to both commensurable and incommensurable magnitudes. It may be expressed symbolically thus—

The ratio of A to B is the same as the ratio of C to D,

when $mA >$, or $=$, or $< nB$,

according as $mC >$, or $=$, or $< nD$,

whatever numbers m and n may be.

In this definition A and B must be magnitudes of the same kind, and so must C and D. But A and C need not be of the same kind. All Euclid's treatment of Proportion is based on this definition; it is therefore most important.

7. Notice that, according to Def. 5, Proportion consists of four terms. When three terms are in continued proportion, there are really four terms; but the third term is the same as the second.

In Algebra we learn that quantities which are in continued proportion are also in geometrical progression, and that the mean proportional between two quantities is the same as the geometrical mean. (See *Tutorial Algebra*, Part I., § 456.)

8. If four magnitudes be in proportion, the first and fourth are called *the extremes*, and the second and third are called *the means;* and the fourth term is said to be a *fourth proportional* to the other three terms.

Thus, if $A : B :: C : D$, A and D are the extremes, B and C are the means, and D is a fourth proportional to A, B, C.

9. If three magnitudes be in continued proportion, the first and third are called *the extremes*, and the second is called a *mean proportional* between the first and third, or simply the *mean;* and the third term is said to be a *third proportional* to the other two terms.

Thus, if $A : B :: B : C$, A and C are the extremes, B is a mean proportional between A and C, or simply the mean, and C is a third proportional to A and B.

10. If a number of magnitudes be in continued proportion,
(1) the ratio of the first to the third is said to be the *duplicate ratio* of the first to the second,
and (2) the ratio of the first to the fourth is said to be the *triplicate ratio* of the first to the second.

Thus, if $A : B :: B : C :: C : D$,
(1) $A : C$ is the duplicate ratio of $A : B$,
(2) $A : D$ is the triplicate ratio of $A : B$.

11. Where there are any number of magnitudes of the same kind, the first is said to have to the last the ratio which is *compounded* of the ratio of the first to the second and of the ratio of the second to the third; and so on to the last magnitude.

Thus, (1) if A, B, C, D be four magnitudes of the same kind, then $A : D$ is the ratio compounded of the ratio $A : B$ and of the ratio $B : C$ and of the ratio $C : D$.

Also, (2) if
$$A : B :: K : L,$$
and $$B : C :: M : N,$$
and $$C : D :: P : Q,$$
then $A : D$ is said to be compounded of the ratios $K : L$, $M : N$, and $P : Q$.

(See Book VI., Prop. 23.)

10 and 11. Either (1) or (2) may be used for compounding ratios. In VI. 23 form (2) is used.

The following definitions and examples taken from *Tutorial Algebra*, Part I., § 388, will probably help the student to understand Euclid.

(i.) *When two ratios are multiplied together, they are said to be compounded.*

Thus $ab : cd$ is said to be compounded of $a : c$ and $b : d$.

The ratio compounded of $a : b$, $b : c$, $c : d$, and $d : e$ is $abcd : bcde$, which is equal to $a : e$.

(ii.) *The ratios $a : b$ and $b : a$ are said to be reciprocal to each other.*

When $a : b$ and $b : a$ are compounded, $ab : ba$ or $1 : 1$ is the result.

(iii.) *When a ratio is compounded with itself, the resulting ratio is called the duplicate ratio of the ratio compounded.*

Thus $a^2 : b^2$ is the duplicate ratio of $a : b$, because it is obtained by compounding $a : b$ with $a : b$.

This algebraical definition can easily be shown to agree with Euclid's definition of duplicate ratio.

For, if $a : b :: b : c$,

$$\therefore \quad \frac{a}{b} = \frac{b}{c};$$

$$\therefore \quad \frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c};$$

$$\therefore \quad \frac{a^2}{b^2} = \frac{a}{c}. \qquad \text{Q. E. D.}$$

Euclid also gives a definition, numbered 7 by him, to teach us how to distinguish the greater of two unequal ratios from the smaller. It is as follows:—"When, of the equimultiples of four magnitudes, taken as in the fifth [= fourth in this book] definition, the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater then the multiple of the fourth, then the first is said to have to the second a greater ratio than the third has to the fourth, and the third is said to have to the fourth a less ratio than the first has to the second." This definition is not required for Book VI.

THEOREMS OF BOOK V. REFERRED TO IN BOOK VI.

(N.B.—The Propositions in Book V. are all Theorems.)

PROP. 7.—(i.) **Equal magnitudes have the same ratio to the same magnitude**; *and conversely* (ii.).

If $A = B$,
then (i.) $A : C :: B : C$,
and (ii.) $C : A :: C : B$.

PROP. 9.—(i.) **Magnitudes which have the same ratio to the same magnitude are equal**; *and conversely* (ii.).

(i.) If $A : C :: B : C$,
then $A = B$.
(ii.) If $C : A :: C : B$,
then $A = B$.

PROP. 11.—**Ratios that are equal to the same ratio are equal to one another.**

If $A : B :: P : Q$,
and $C : D :: P : Q$,
then $A : B :: C : D$.

PROP. 12.—**If any number of ratios are equal, all the magnitudes being of the same kind, as one of the antecedents is to its consequent so is the sum of all the antecedents to the sum of all the consequents.**

If $A : B :: C : D :: E : F$,
then $A : B :: A+C+E : B+D+F$. **Addendo.**

PROP. 15.—**Magnitudes have the same ratio to one another that their equimultiples have.**

$A : B :: mA : mB$.

PROP. B. — If four magnitudes are proportionals, they are also proportionals when taken inversely.

If $A : B :: C : D$,

then $B : A :: D : C$. **Invertendo.**

PROP. 16. — If four magnitudes of the same kind are proportionals, they are also proportionals when taken alternately.

If $A : B :: C : D$,

then $A : C :: B : D$. **Alternando.**

PROP. 17. — If four magnitudes are proportionals, the difference between the first antecedent and its consequent is to that consequent as the difference between the second antecedent and its consequent is to that consequent.

If $A : B :: C : D$,

then $A \sim B : B :: C \sim D : D$. **Dividendo.**

PROP. 18.—If four magnitudes are proportionals, the sum of the first antecedent and its consequent is to that consequent as the sum of the second antecedent and its consequent is to that consequent.

If $A : B :: C : D$,

then $A + B : B :: C + D : D$. **Componendo.**

PROP. 22.—Ratios compounded of equal ratios are equal.

If $A : B :: P : Q$,

and $B : C :: Q : R$,

then $A : C :: P : R$. **Ex aequali.**

PROP. 24.— If $A : B :: C : D$,

and $E : B :: F : D$,

then $A + E : B :: C + F : D$.

Additional Propositions.

(i.) **If two ratios are equal, their duplicate ratios are equal.**

Let $A : B :: C : D$;

then shall the duplicate ratio of $A : B$ be equal to the duplicate ratio of $C : D$.

Take P a third proportional to A and B, so that

$$A : B :: B : P;$$

and take Q a third proportional to C and D, so that

$$C : D :: D : Q.$$

Proof.—Because $A : B :: C : D$,

therefore $B : P :: D : Q$;

therefore, *ex aequali*, $A : P :: C : Q$.

But $A : P$ and $C : Q$ are respectively the duplicate ratios of $A : B$ and $C : D$;

therefore the duplicate ratio of $A : B$ is equal to the duplicate ratio of $C : D$. Q. E. D.

(ii.) **If the duplicate ratios of two ratios are equal, the ratios themselves are equal.**

Let the duplicate ratio of $A : B$ be equal to the duplicate ratio of $C : D$;

then shall $A : B :: C : D$.

Take X such that $A : B :: C : X$;

therefore, *invertendo*, $B : A :: X : C$.

Also, since the duplicate ratios are equal,

therefore $A : P :: C : Q$;

therefore, *ex aequali*, $B : P :: X : Q$.

But $A : B :: B : P$;

therefore $A : B :: X : Q$;

therefore $C : X :: X : Q$.

But $C : D :: D : Q$;

therefore $X = D$;

therefore $A : B :: C : D$. Q. E. D.

Notes on the Propositions in Book V.

The Fifth Book of Euclid deals with Ratio and Proportion, and the Sixth Book to a large extent depends on the Fifth Book. The Fifth Book is difficult, and it is therefore often omitted. Such properties of Ratio and Proportion as are required in the Sixth Book the student is generally familiar with in Algebra, and he assumes their truth in Geometry. This assumption, however, is not always possible, and is seldom satisfactory, as the following remarks will show:—

Arithmetic and Algebra deal with quantities expressed by numbers, and these numbers do not vary continuously, and the quantities can generally be expressed in terms of some common unit. Now Plane Geometry deals with the magnitudes of lines, angles, and areas which vary continuously, and they often cannot be expressed in terms of a common unit. It is therefore necessary to establish the treatment of Geometrical Ratio and Proportion on a different basis from that on which Arithmetic and Algebra rest.

Euclid bases his treatment of Ratio and Proportion on Definition 4 (see page 4). The student is advised to master this definition even if he omits everything else in Book V. He will find the definition used in Props. 1 and 33 in Book VI. The rest of Book V. he may leave for study at a later period.

We have given the enunciations and statements of the more important theorems in Book V., and they will be found to include all that are required in Book VI. These propositions are marked with Euclid's numbers.

Props. 7 and 9.—These are converse propositions.

Prop. 11.—A series of ratios that are all the same may be written thus:—

$$A : B :: C : D :: E : F :: P : Q, \quad \&c.,$$

or thus:—

$$A : B = C : D = E : F = P : Q, \quad \&c.$$

Prop. B.—This proposition was added by Simson.

Prop. 17.—This proposition may be stated thus:—

If $A+B : B :: C+D : D,$

then $A : B :: C : D.$

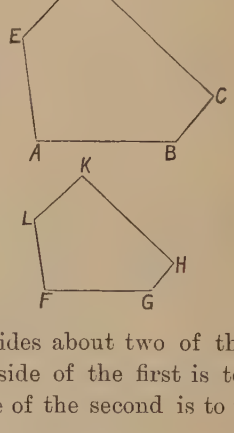
In this form it will be seen to be the converse of Prop. 18.

Prop. 18.—The student must not confuse Componendo in Proportion, which is of the nature of addition, with Compounding Ratios, which in Algebra depends on multiplication.

BOOK VI.

ON PROPORTION APPLIED TO GEOMETRY.

DEFINITIONS.



1. *Equiangular rectilineal figures* are such as have their angles, taken in order, equal, each to each.

2. *Similar rectilineal figures* are such as are equiangular and have the sides about their equal angles proportional, the corresponding sides being homologous.

3. Two figures are said to have the sides about two of their angles *reciprocally proportional* when a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first.

4. A straight line is said to be cut in extreme and mean ratio when the whole is to the greater segment as the greater segment is to the less.

5. The *altitude* of a triangle is the straight line drawn from its vertex perpendicular to the base.

The *altitude* of a parallelogram is the straight line drawn from any point on one side of the parallelogram perpendicular to the opposite side.

6. Two similar rectilineal figures are said to be *similarly described* with respect to two of their sides when these sides are corresponding or homologous sides. (See *Def. 6*, page 4.)

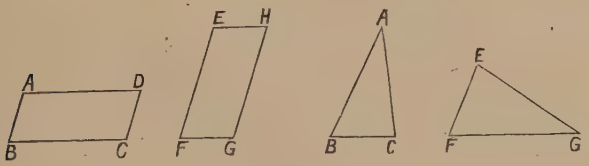
[For Additional Definitions, see page 17.]

Notes on the Definitions.

1. The figures *ABCDE* and *FGHKL* are *equiangular* if the angles *A*, *B*, *C*, *D*, *E* are equal to the angles *F*, *G*, *H*, *K*, *L*, respectively, each to each, that is, $\angle A = \angle F$, $\angle B = \angle G$, and so on.

Pairs of angles which are equal, such as *A* and *F*, *B* and *G*, &c., are called *corresponding angles*; and pairs of sides which join corresponding angles, such as *AB* and *FG*, *BC* and *GH*, &c., are called *corresponding sides*.

2. In the similar figures *ABCDE* and *FGHKL* on page 12, $AB : BC :: FG : GH$, also $BC : CD :: GH : HK$, &c.



3. The parallelograms *ABCD* and *EFGH* and the triangles *ABC* and *EFG* have their sides about the angles *B* and *F* reciprocally proportional if $AB : EF :: FG : BC$.

4. In the figure, $AB : AC :: AC : CB$. (See Book II., Prop. 11.)

5. The *altitude* of any figure is its perpendicular height. Any side of a triangle may be considered as its base; a triangle therefore has three altitudes. Similarly, a parallelogram has two altitudes. These altitudes may or may not be equal to one another.

In an isosceles triangle the base is usually taken to be the side which is not equal to either of the other sides, and the altitude of an isosceles triangle is usually taken to be the perpendicular from the vertex on this unequal side.

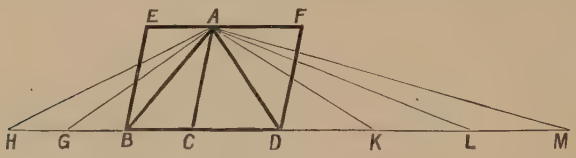
In a square or rectangle the altitude is equal to one of the sides of the figure. In an equilateral triangle the three altitudes are all equal and are the same as the medians. In an isosceles triangle the altitude is the same as one of the medians.

6. The figures of Defs. 1 and 2 are similarly described with respect to the sides *AB* and *FG*, or with respect to *GH* and *BC*, &c.

PROP. 1. — *Theorem.* — **The areas of triangles or parallelograms of the same altitude are proportional to the lengths of their bases.**

Let ABC and ACD be two triangles, and EC and CF two parallelograms of the same altitude, namely, the perpendicular from A to BC;

then shall (1) the area of $\triangle ABC$: the area of $\triangle ACD$ :: BC : CD, and (2) the area of $\square EC$: the area of $\square CF$ :: BC : CD.



(1) **Construction.**—Produce BD both ways;
and from CB produced cut off any number of parts BG, GH, each equal to CB;
and from CD produced cut off any number of parts DK, KL, LM, each equal to CD.
Join AH, AG, AK, AL, AM.

Proof.—Because $\triangle$'s ABC, AGB, AHG are on equal bases and between the same parallels,
therefore $\triangle$'s ABC, AGB, AHG are equal in area; *I. 38*
therefore $\triangle AHC$ is the same multiple of $\triangle ABC$ that HC is of BC.
Similarly $\triangle ACM$ is the same multiple of $\triangle ACD$ that CM is of CD.
And, if $HC = CM$,
$\triangle AHC = \triangle ACM$;
and, if HC is greater than CM,
$\triangle AHC$ is greater than $\triangle ACM$;
and, if HC is less than CM,
$\triangle AHC$ is less than $\triangle ACM$.
Then, since there are four magnitudes, namely, $\triangle$'s ABC, ACD and the bases BC, CD; and of the first and third any equi-

multiples have been taken, namely, $\triangle AHC$ and the base HC;
and of the second and fourth any equimultiples have been taken,
namely, $\triangle ACM$ and the base CM;
and, since it has been shown that $\triangle AHC$ is greater than, equal to, or less than $\triangle ACM$, according as HC is greater than, equal to, or less than CM,
therefore the four original magnitudes are proportionals; *V. Def. 4*
therefore the area of $\triangle ABC$: the area of $\triangle ACD :: BC : CD$.
Q. E. D.

(2) Again, because
the area of $\triangle ABC$: the area of $\triangle ACD :: BC : CD$,
and $\square EC$ is double of $\triangle ABC$,
and $\square CF$ is double of $\triangle ACD$,
and magnitudes have the same ratio that their equimultiples have, *V. 15*
therefore $\square EC : \square CF :: BC : CD$. Q. E. D.

Exercises on Proposition 1.

1. Prove that the areas of triangles or parallelograms of equal altitudes are proportional to the lengths of their bases.

2. If straight lines AO, BO, CO be drawn from the vertices of a triangle ABC to any point O within the triangle, and if AO produced cuts BC in L, prove that $\triangle AOB : \triangle AOC :: BL : LC$.

3. $ABCD$ is a quadrilateral, and its diagonals AC and BD intersect in O. Show that $\triangle AOB : \triangle BOC :: \triangle AOD : \triangle DOC$.

4. ABC is a triangle. Find a point P in ABC such that the triangles PAB, PBC, and PCA are all equal.

5. The diagonals of a trapezium, two of whose sides are parallel and one of them double of the other, cut each other at a point of trisection.

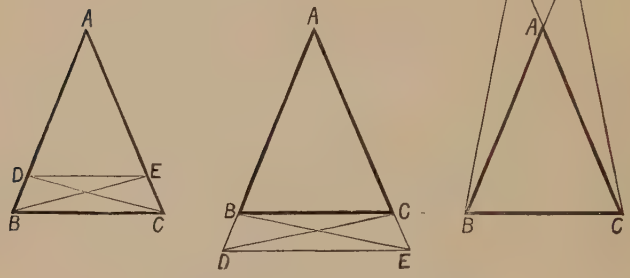
6. The sum of the perpendiculars drawn from any point within an equilateral triangle is constant.

PROP. 2. — *Theorem.* — (i.) **If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those sides produced, proportionally;**

and, conversely, (ii.) **If the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.**

(i.) Let the straight line DE be drawn parallel to BC, one of the sides of $\triangle ABC$;

then shall $BD : DA :: CE : EA$.



Construction.—Join BE, CD.

Proof.—Because $\triangle BDE = \triangle CDE$, being on the same base DE and between the same parallels DE and BC, *I. 37*

and ADE is another triangle,

therefore $\triangle BDE : \triangle ADE :: \triangle CDE : \triangle ADE$. *V. 7*

But $\triangle BDE : \triangle ADE :: BD : DA$, *VI. 1*

and $\triangle CDE : \triangle ADE :: CE : EA$; *VI. 1*

therefore $BD : DA :: CE : CA$. *V. 11*

Q. E. D.

(ii.) Conversely, Let $BD : DA :: CE : EA$,

and join DE;

then shall DE be parallel to BC.

Construction.—Join BE, CD.

Proof.—Because $BD : DA :: CE : EA$, *Hyp.*

and $BD : DA :: \triangle BDE : \triangle ADE$, *VI. 1*

and $CE : EA :: \triangle CDE : \triangle ADE$, *VI. 1*

therefore $\triangle BDE : \triangle ADE :: \triangle CDE : \triangle ADE$; *V. 11*

therefore $\triangle BDE = \triangle CDE$, *V. 9*

and equal triangles on the same base and on the same side of it are between the same parallels; *I. 39*

therefore DE is parallel to BC. Q. E. D.

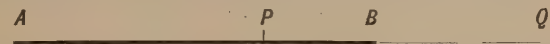
Exercises on Proposition 2.

1. In the figure of VI. 2 prove that $AB : AD :: AC : AE$.

2. If two sides of a quadrilateral be parallel, any straight line drawn parallel to these two sides will cut the other sides proportionally.

Additional Definitions.

7. A straight line is said to be *divided harmonically* when it is divided internally and externally in the same ratio.



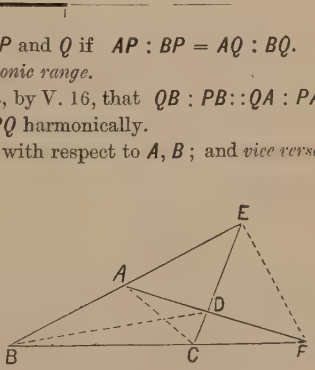
The line AB is divided harmonically at P and Q if $AP : BP = AQ : BQ$.

The points A, P, B, Q are called a *harmonic range*.

From the above proportion, it is evident, by V. 16, that $QB : PB :: QA : PA$, that is, B and A divide the straight line PQ harmonically.

The points P, Q are *harmonic conjugates* with respect to A, B; and *vice versa*.

8. A *complete quadrilateral* is the figure formed by four infinite straight lines.



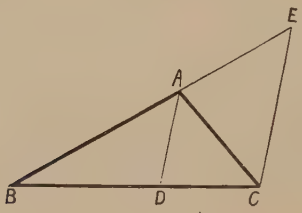
The four lines intersect in six *vertices*, and three *diagonals* can be drawn by joining opposite vertices. In the figure A, B, C, D, E, and F are the six vertices, and AC, BD, and EF are the three diagonals.

PROP. 3. — *Theorem.* — (i.) **If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have to one another the same ratio as the other sides of the triangle;**

and, conversely, (ii.) **If the segments of the base have to one another the same ratio as the other sides of the triangle, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.**

(i.) Let ABC be a triangle, and let the vertical angle BAC be bisected by the straight line AD, which also cuts the base at D;

then shall $BD : DC :: BA : AC$.



Construction.—Through C draw CE parallel to AD, to meet BA produced in E.

Proof.—Because AD is parallel to CE,

therefore $\angle BAD =$ the interior opposite $\angle AEC$, *I. 29*

and also $\angle DAC =$ the alternate $\angle ACE$. *I. 29*

But $\angle BAD = \angle DAC$, since AD bisects the vertical $\angle BAC$; *Hyp.*

therefore $\angle AEC = \angle ACE$;

therefore $AC = AE$. *I. 6*

Again, because AD is parallel to CE, one of the sides of $\triangle BCE$,

therefore $BD : DC :: BA : AE$; *VI. 2*

but $AE = AC$; *Proved*

therefore $BD : DC :: BA : AC$. Q. E. D.

(ii.) Conversely, let $BD : DC :: BA : AC$,
and join AD;
then shall AD bisect the vertical angle BAC.

Construction.—As before, through C draw CE parallel to AD, to meet BA produced in E.

Proof.—Because $BD : DC :: BA : AC$, *Hyp.*
and because AD is parallel to CE, one of the sides of $\triangle BCE$,
therefore $BD : DC :: BA : AE$; *VI. 2*
therefore $BA : AC :: BA : AE$;
therefore $AC = AE$; *V. 9*
therefore $\angle AEC = \angle ACE$. *I. 5*
And, because AD is parallel to CE,
therefore the interior $\angle AEC$ = the exterior $\angle BAD$, *I. 29*
and $\angle ACE$ = the alternate $\angle DAC$; *I. 29*
therefore $\angle BAD = \angle DAC$;
therefore AD bisects the vertical angle BAC. Q. E. D.

Exercises on Propositions 2 and 3.

1. Prove that the straight line drawn through the middle point of one of the sides of a triangle parallel to the base will bisect the other side of the triangle.

2. ABC is any triangle, and O is any given point. Join OA, OB, and OC, and on OA, OB, and OC take the points P, Q, and R such that QR is parallel to BC and RP to CA. Prove that PQ is parallel to AB.

3. Prove that there is only one point which divides a given straight line internally in a given ratio.

4. Assuming Prop. 3, show how to trisect a given line AB.

5. ABC is a triangle, and the side BC is bisected at P, and the angles APB and APC are bisected by the lines PQ and PR meeting AB and AC in Q and R, respectively. Prove that QR is parallel to BC.

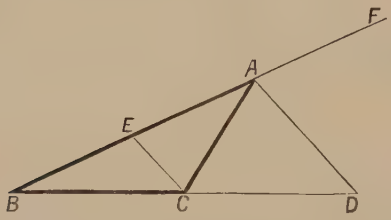
6. If the line joining the points of intersection of two sides of a triangle with the bisectors of the opposite angles is parallel to the base, the triangle is isosceles.

PROP. A.—*Theorem.*—(i.) **If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line which also cuts the base produced, the segments between the bisector and the extremities of the base shall have to one another the same ratio as the other sides of the triangle;**

and, conversely, **(ii.) If the segments of the base produced have to one another the same ratio as the other sides of the triangle, the straight line drawn from the vertex to the point of section shall bisect the exterior vertical angle of the triangle.**

(i.) Let ABC be a triangle, having the side BA produced to F, and let the exterior angle CAF be bisected by the straight line AD, which also cuts the base produced at D;

then shall $BD : CD :: BA : AC$.



Construction.—Through C draw CE parallel to AD, to meet BA in E.

Proof.—Because AD is parallel to CE,

$\therefore$ $\angle FAD$ = the interior opposite $\angle AEC$, *I. 29*

and also $\angle DAC$ = the alternate $\angle ACE$. *I. 29*

But $\angle FAD = \angle DAC$, since AD bisects the exterior $\angle CAF$; *Hyp.*

$\therefore$ $\angle AEC = \angle ACE$;

$\therefore$ $AC = AE$. *I. 6*

Again, because AD is parallel to CE, one of the sides of $\triangle BCE$,

$\therefore$ $BD : CD :: BA : AE$; *VI. 2*

but $AE = AC$; *Proved*

$\therefore$ $BD : CD :: BA : AC$. Q. E. D.

(ii.) Conversely, let $BD : CD :: BA : AC$,
and join AD;
then shall AD bisect the exterior vertical $\angle CAF$.

Construction.—As before, through C draw CE parallel to AD to meet BA in E.

Proof.—Because $BD : CD :: BA : AC$, *Hyp.*
and because AD is parallel to CE, one of the sides of $\triangle BCE$,
$\therefore$ $BD : CD :: BA : EA$; *VI. 2*
$\therefore$ $BA : AC :: BA : EA$;
$\therefore$ $AC = AE$; *V. 9*
$\therefore$ $\angle AEC = \angle ACE$. *I. 5*
And, because AD is parallel to CE,
$\therefore$ the interior $\angle AEC$ = the exterior $\angle FAD$, *I. 29*
and $\angle ACE$ = the alternate $\angle DAC$; *I. 29*
$\therefore$ $\angle FAD = \angle DAC$;
$\therefore$ AD bisects the exterior vertical $\angle CAF$. Q. E. D.

Exercises on Propositions 3 and A.

1. If the vertical angle of a triangle be bisected, both internally and externally, the bisectors divide the base harmonically. (See Def. 7, page 17.)

2. Find the locus of a point which moves so that the ratio of its distances from two given points is constant.

3. A given straight line AB is divided into two parts AC and CB. Show that the locus of a point P, at which AC and CB subtend equal angles, is a circle.

4. The straight lines bisecting one interior and two exterior angles of a triangle are concurrent.

5. Two circles touch one another internally at O, and a chord PQ of the outer circle touches the inner circle at R. Show that
$$OP : OQ :: PR : RQ.$$

6. Show that the centre of the inscribed circle of a triangle divides the bisector of the vertical angle of the triangle into segments which have the same ratio to each other as the base to the sum of the two sides of the triangle.

PROP. 4.—*Theorem.*—**If two triangles be equiangular to one another, they shall be similar:** that is, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.

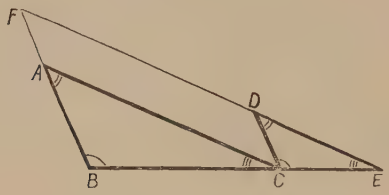
Let ABC and DCE be two triangles which are equiangular to one another, having $\angle ABC$ equal to $\angle DCE$ and $\angle ACB$ equal to $\angle DEC$, and consequently $\angle BAC$ equal to $\angle CDE$;

then the triangle ABC shall be similar to the triangle DCE;

that is, $AB : BC :: DC : CE$,

$BC : AC :: CE : DE$,

and $AB : AC :: DC : DE$.



Construction.—Place $\triangle$'s ABC and DCE so that CE may be contiguous to BC, and in the same straight line with BC.

Then, because $\angle ACB = \angle DEC$, *Hyp.*

add to each the angle ABC;

$\therefore$ the two $\angle$'s ABC and ACB = the two $\angle$'s ABC and DEC;

but the two $\angle$'s ABC and ACB are together less than two right angles, being in the same triangle ABC; *I. 17*

$\therefore$ the two $\angle$'s ABC and DEC are also together less than two right angles;

$\therefore$ BA and ED will meet, if produced, towards A and D; *Ax. 12*

let them be produced and meet in F.

Proof.—(1) Because $\angle ABC = \angle DCE$, *Hyp.*

$\therefore$ FB is parallel to DC; *I. 28*

and, because $\angle ACB = \angle DEC$, *Hyp.*

$\therefore$ AC is parallel to EF; *I. 28*

$\therefore$ $FACD$ is a parallelogram;
$\therefore$ $FA = DC$ and $AC = FD$. *I. 34*

(2) Again, because AC is parallel to FE, a side of $\triangle BFE$,
$\therefore$ $BA : AF :: BC : CE$; *VI. 2*
but $AF = CD$; *Proved*
$\therefore$ $BA : CD :: BC : CE$;
and, alternately, $AB : BC :: DC : CE$. *V. 16*
Again, because CD is parallel to BF, a side of $\triangle BFE$;
$\therefore$ $BC : CE :: FD : DE$; *VI. 2*
but $FD = AC$; *Proved*
$\therefore$ $BC : CE :: AC : DE$;
and, alternately, $BC : AC :: CE : DE$. *V. 16*
And $AB : BC :: DC : CE$; *Proved*
$\therefore$ *ex aequali*, $AB : AC :: DC : DE$. *V. 22*
Q. E. D.

Exercises on Proposition 4.

1. If a man 6 feet high has a shadow 11 feet long, what is the height of a tower whose shadow is 44 yds. long?

2. On a given straight line as base, show how to describe a triangle which shall be similar to a given triangle.

3. AB and CD are two chords of a circle which intersect at the point O within the circle. Prove that the triangles AOD and COB are similar.

4. Show from Ex. 3 that, if the rectangle contained by two straight lines is equal to the rectangle contained by two other straight lines, then the first line is to the third as the fourth is to the second.

5. If a tangent OT be drawn to a circle from a given point O, and also a secant OAB, show that the triangles OAT and OTB are similar, and that OT is a mean proportional between OA and OB.

6. A common tangent to two circles cuts the straight line joining their centres externally or internally in the ratio of the radii of the circles.

7. Inscribe an equilateral triangle in a given triangle so as to have one side parallel to a side of the given triangle.

8. Prove Prop. 4 by the method of superposition.

Alternative Proof.

PROP. 4.—*Theorem.*—**If two triangles be equiangular to one another, they shall be similar**: that is, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.

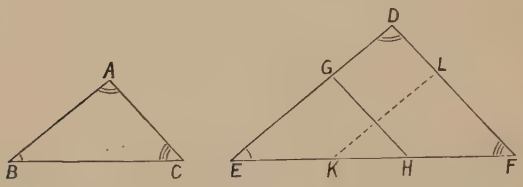
Let ABC and DEF be two triangles which are equiangular to one another, having $\angle ABC$ equal to $\angle DEF$, and $\angle ACB$ equal to $\angle DFE$, and consequently $\angle BAC$ equal to $\angle EDF$;

then the triangle ABC shall be similar to the triangle DEF;

that is, $AB : BC :: DE : EF$,

$BC : AC :: EF : DF$,

and $AB : AC :: DE : DF$.



Construction.—If the side AB is equal to the side DE, then, by I. 26, the triangles ABC and DEF will be equal in all respects, and therefore similar.

If AB is not equal to DE, one of them must be the greater; suppose DE is greater than AB, and from DE cut off GE equal to AB; *I. 3*

from EF cut off EH equal to BC;

and join GH.

Proof.—In the two $\triangle$'s ABC, GEH,

because $\begin{cases} AB = GE, \\ BC = EH, \\ \text{and } \angle ABC = \angle GEH, \end{cases}$ *Hyp.*

$\therefore$ by I. 4, the triangles are equal in all respects; *I. 4*

that is, $\angle BAC = \angle EGH$.

But $\angle BAC = \angle EDF$; *Hyp.*

$\therefore \angle EGH = \angle EDF$;

$\therefore$ GH is parallel to DF. *I. 28*

(2) Again, because DF is parallel to GH, a side of $\triangle GEH$,

$\therefore GE : DE :: EH : EF$. *VI. 2*

But $GE = AB$ and $EH = BC$;

$\therefore AB : DE :: BC : EF$;

and, alternately, $AB : BC :: DE : EF$. *V. 16*

Q. E. D.

Similarly, by making FK and FL equal to CB and CA, respectively, it can be proved that

$BC : AC :: EF : DF$;

$\therefore$ *ex aequali*, $AB : AC :: DE : DF$. *V. 22*

Q. E. D.

Exercises.

1. Every straight line which is drawn from the vertex of a given triangle to the base is bisected by the straight line which joins the middle points of the other two sides of the triangle.

2. A quadrilateral $ABCD$ is inscribed in a circle, and the two sides BA and CD are produced to meet in O; prove that the triangles OAD and OBC are similar.

3. State, and prove, the converse of Ex. 2.

4. ABC is any triangle, and the lines AD, BE, CF are drawn making the angles BAD, CBE, ACF all equal to one another; prove that the lines either meet in a point or else form a triangle similar to the triangle ABC.

5. In the figure of the complete quadrilateral on page 17 let AD and BC meet in G; join EG and produce it to meet BC in H; prove that B, H, C, F is a harmonic range.

PROP. 5.—*Theorem.*—**If the sides of two triangles, taken in order, about each of their angles, be proportionals, the triangles shall be equiangular to one another, having those angles equal which are opposite to the homologous sides.**

Let ABC and DEF be two triangles, having the sides about each of their angles proportionals, that is,

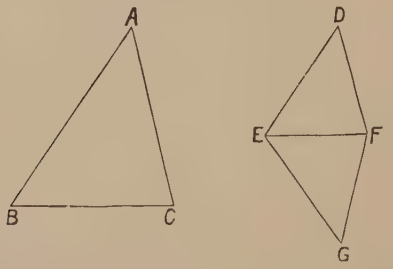
$$AB : BC :: DE : EF,$$

and $$BC : CA :: EF : FD,$$

and, consequently, *ex aequali*,

$$AB : CA :: DE : FD;$$

then the triangles ABC and DEF shall be equiangular to one another.



Construction.—At the point E in the straight line EF, make the angle FEG equal to the angle ABC; *I. 23*

and at the point F in the straight line EF make the angle EFG equal to the angle ACB; *I. 23*

then the remaining angle EGF = the remaining angle BAC. *I. 32*

Proof.—Because $\triangle EFG$ is equiangular to $\triangle ABC$,

$\therefore$ $\triangle EFG$ is similar to $\triangle ABC$; *VI. 4*

$\therefore$ $AB : BC :: GE : EF.$

But $AB : BC :: DE : EF$; *Hyp.*

$\therefore$ $GE : EF :: DE : EF$; *V. 1*

$\therefore$ $GE = DE.$ *V. 9*

Similarly, it can be proved that $GF = DF$.

Then, in the two $\triangle$'s GEF, DEF,

because $\left\{\begin{array}{l} GE = DE, \\ \text{and } EF \text{ is common to both triangles,} \\ \text{and the base } GF = \text{the base } DF; \end{array}\right.$

$\therefore$ by I. 8, the triangles are equal in all respects; *I. 8*

that is, $\angle GEF = \angle DEF$,

and $\angle GFE = \angle DFE$,

and $\angle EGF = \angle EDF$.

But $\angle GEF = \angle ABC$; *Constr.*

$\therefore$ $\angle DEF = \angle ABC$.

Similarly, $\angle EFD = \angle ACB$;

and $\therefore$ the remaining $\angle EDF$ = the remaining $\angle BAC$;

$\therefore$ $\triangle ABC$ is equiangular to $\triangle DEF$. Q. E. D.

Miscellaneous Exercises.

1. Two isosceles triangles whose vertical angles are equal have their altitudes proportional to their bases.

2. If AX, BY, CZ be perpendicular to the sides BC, CA, AB of the triangle ABC, then YZ is divided harmonically by AX and BC.

3. In the figure of VI. 2, if BE and CD intersect in O, show that the triangle OBA is equal to the triangle OCA.

4. On a given base describe a triangle having a given vertical angle and the remaining sides of the triangle in a given ratio.

5. The sides of a triangle are 4, 5, and 6 inches long. Find the lengths of the segments into which the bisector of the largest angle divides the opposite side.

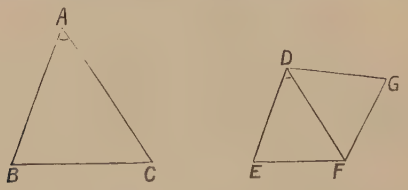
6. ABC is any triangle, and P is a point in AB. From P draw a straight line to meet AC produced in Q, such that PQ is bisected by BC.

7. If P be any point on a circle whose diameter is AB, show that the perpendicular from P on AB and the tangent to the circle at P divide AB harmonically.

8. Show that the median AP of the triangle ABC bisects every line which is drawn parallel to the base BC and terminated by AB and AC.

PROP. 6. — *Theorem.* — **If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be similar.**

Let ABC and DEF be two triangles, having $\angle BAC$ equal to $\angle EDF$
and let $BA : AC :: ED : DF$;
then the triangle ABC shall be similar to the triangle DEF.



Construction.—At the point D in the straight line DF make the angle FDG equal to the angle BAC or EDF, *I. 23*
and at the point F in the straight line DF make the angle DFG equal to the angle ACB; *I. 23*
then the remaining angle DGF = the remaining angle ABC. *I. 32*

Proof.—Because $\triangle ABC$ is equiangular to $\triangle DGF$,
$\therefore$ $\triangle ABC$ is similar to $\triangle DGF$; *VI. 4*
$\therefore$ $BA : AC :: GD : DF$.
But $BA : AC :: ED : DF$; *Hyp.*
$\therefore$ $GD : DF :: ED : DF$;
$\therefore$ $GD = ED$. *V. 9*
Then, in the two $\triangle$'s GDF, EDF,
because $\begin{cases} GD = ED, \\ \text{and } DF \text{ is common to both triangles,} \\ \text{and } \angle GDF = \angle EDF, \end{cases}$ *Constr.*
$\therefore$ by I. 4, the triangles are equal in all respects; *I. 4*
that is, the triangle GDF is equiangular to the triangle EDF.
But the triangle GDF is equiangular to the triangle BAC;
$\therefore$ the triangle EDF is equiangular to the triangle BAC;
$\therefore$ the triangle ABC is similar to the triangle DEF. Q. E. D.

Notes on Propositions 1 to 5.

The student is apt to find some of the propositions in Book VI. difficult because the propositions of Book V. are not usually studied, and the difficult propositions in Book VI. will always be found to depend on some part of Book V. The student therefore is advised, as he advances in Book VI., to turn back to Book V. and to read over that part of Book V. which is referred to in any particular proposition in Book VI. More especially will such references be useful in Propositions 1, 19, and 23 of Book VI.

Proposition 1.—Before reading this proposition the student should make himself familiar with Euclid's definition of proportion in Book V. (see Def. 4, page 4). The proof depends on that definition.

Proposition 2.—This is a very important proposition, and is often used to prove other propositions (see VI. 3, A, 4, 9, 10, 11, 12, 24). There are three cases, according as the line ***DE*** cuts the sides ***AB*** and ***AC*** unproduced, or produced towards ***BC***, or produced towards ***A***. The proofs of both (i.) and (ii.) depend on VI. 1.

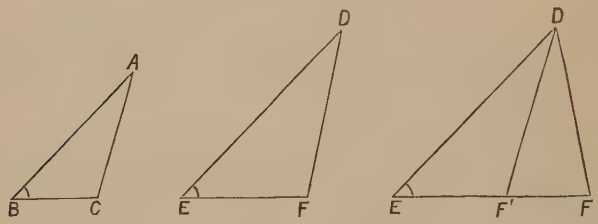
Propositions 3 and A.—These are two cases of the same proposition, and both may be enunciated at once, thus :—*If the vertical angle of a triangle be bisected either internally or externally by a straight line which also cuts the base or the base produced, then the segments of the base shall have to one another the same ratio as the other sides of the triangle; and conversely*, &c. Proposition A was added by Simson. Both propositions are proved by VI. 2.

Proposition 4.—Homologous sides are sides which are the antecedents or the consequents of the ratios (see V., Def. 6). Propositions 4, 5, and 6 might easily be proved by the method of superposition. The proof of Prop. 4 depends on VI. 2. The Alternative Proof avoids placing the two triangles side by side; it also depends on VI. 2.

Proposition 5.—This proposition is the converse of VI. 4. Similar triangles have two properties, viz. : they are equiangular, and they have their sides taken in order proportional. If either of these properties is given, the other follows. This is a peculiar property of triangles, and is not true of other figures. For example, a square and a rectangle are equiangular, but their sides taken in order are not proportional. The words *taken in order* are important; for triangles may have their sides about each of their angles proportional, but not when *taken in order*, and then the triangles are not necessarily equiangular.

PROP. 7.—*Theorem.*—**If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite the equal angles are homologous, then the triangles shall have their third angles equal or supplementary; and in the former case the triangles will be similar.**

Let ABC and DEF be two triangles, having $\angle ABC$ equal to $\angle DEF$
and let $AB : AC :: DE : DF$;
then the third $\angle ACB$ shall be either (1) equal or (2) supplementary to the third $\angle DFE$; and in the former case the triangles shall be similar.



CASE I.—If $\angle BAC = \angle EDF$;
then the third $\angle ACB =$ the third $\angle DFE$; *I. 32*
$\therefore$ the triangles ABC and DEF are equiangular and therefore similar. *VI. 4* Q. E. D.

CASE II.—If $\angle BAC$ is not equal to $\angle EDF$,
then one of them must be the greater.
Let $\angle EDF$ be greater than $\angle BAC$.

Construction.—At the point D in the straight line ED make the angle EDF' equal to the angle BAC, and let DF' meet EF in F'. *I. 23*

Proof.—Because the triangles BAC and EDF' are equiangular,
$\therefore$ they are similar.
$\therefore$ $AB : AC :: DE : DF'$. *VI. 4*

But $AB : AC :: DE : DF$; *Hyp.*

$\therefore\ DF = DF'$;

$\therefore\ \angle DF'F = \angle DFF'$; *I. 6*

$\therefore$ the $\angle$'s $DF'E$ and DFE are supplementary. *I. 13*

But $\angle DF'E = \angle ACB$;

$\therefore\ \angle$'s ACB and DFE are supplementary. Q. E. D.

Corollary.—If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, and if the third angles be either (1) both acute, or (2) both obtuse, or (3) if one of them be a right angle, then the two triangles will be similar.

For the third angles cannot be supplementary unless one is obtuse and the other is acute; and, as this is impossible by (1), (2), or (3), the triangles must be equiangular and therefore similar by the above proposition.

Notes on Similar Triangles.

Propositions 4 to 7 show that two triangles will be similar if the following quantities are equal in each :—

(i.) Two angles (the third angles being equal by I. 32). *VI. 4*

(ii.) Two of the ratios of the three sides (the third ratio being equal *ex aequali*). *VI. 5*

(iii.) One angle and the ratio of the sides containing the angle. *VI. 6*

(iv.) One angle and the ratio of the sides about another angle, provided the third angles be both acute or both obtuse, or one of them a right angle. *VI. 7*

In each case the necessary conditions of similarity of two triangles are two in number. Euclid gives more conditions than are absolutely necessary.

It is useful to compare with the above cases of the Similarity of two triangles the cases of the Equality of two triangles in Book I. (See Bk. I., p. 53.)

(i.) VI. 4 corresponds to I. 26 and Case V. (page 53).

(ii.) VI. 5 ,, ,, I. 8.

(iii.) VI. 6 ,, ,, I. 4.

(iv.) VI. 7 ,, ,, Case VI. (page 53). (See page 101, Ex. 2.)

PROP. 8.—*Theorem.*—**In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.**

Let ABC be a right-angled triangle, having $\angle BAC$ a right angle;
and from A draw AD perpendicular to BC;
then the triangles ABD and ADC shall be similar to the whole triangle ABC, and to each other.

Proof.—Because, in the two triangles ABD and ABC,
$\angle ADB = \angle BAC$, being right angles,
and the angle at B is common to both triangles,
$\therefore$ the third $\angle BAD =$ the third $\angle ACB$; *I. 32*
$\therefore$ the triangles ABD and ABC are equiangular;
$\therefore$ the triangles ABD and ABC are similar. *VI. 4*
Q. E. D.

In the same manner it may be proved that the triangles ADC and ABC are equiangular and therefore similar.
And, since each of the triangles ABD and ADC is equiangular to the whole triangle ABC,
$\therefore$ the triangles ABD and ADC are equiangular to each other;
and $\therefore$ the triangles ABD and ADC are similar to each other. *VI. 4*
Q. E. D.

Corollary 1.—The perpendicular AD is a mean proportional between BD and DC, the segments of the base.
For, by similar triangles, $BD : AD :: AD : DC$.

Corollary 2.—The side AB is a mean proportional between BC and BD.

For, by similar triangles, $BC : AB :: AB : BD$.

Corollary 3.—The side AC is a mean proportional between BC and CD.

For, by similar triangles, $BC : AC :: AC : CD$.

NOTE.—Before attempting to solve any rider on similar triangles or similar rectilinear figures, the student should mark corresponding angles and sides in the same way as equal angles and sides were marked in Book I. (see page 53); and in difficult riders it will simplify matters if the student draws the similar figures separately and then marks corresponding angles and sides, as in the figures of I. 5 and 6.

MISCELLANEOUS EXERCISES ON PROPOSITIONS 1 TO 8.

1. If two straight lines AOD, BOC cut one another internally or externally in O, so that $OA : OB :: OC : OD$, prove that A, B, C, D are concyclic.

2 Prove that the medians of a triangle divide each other into segments that are in the ratio of 1 : 2.

3. ABC is a triangle, and a perpendicular is drawn from A to the opposite side meeting it at D between B and C. Show that, if AD is a mean proportional between BD and DC, the angle BAC is a right angle.

4 In a right-angled triangle the perpendicular on the hypotenuse is a fourth proportional to the hypotenuse and the other two sides.

5. If AD, BE be drawn from the vertices A and B of the triangle ABC perpendicular to the opposite sides, and if the perpendiculars cut each other in O, prove that $BD : DO :: AD : DC$.

6. By means of Ex. 5, prove that the three perpendiculars drawn from the angular points of a triangle to the opposite sides all pass through the same point.

7. If two circles touch one another externally, prove that the portion of the common tangent intercepted between its two points of contact is a mean proportional between the diameters of the circles.

8. Prove that triangles and parallelograms which are to one another as their altitudes stand on equal bases.

9. If two circles touch each other either externally or internally, any two straight lines drawn through the point of contact will be cut proportionally by the circumferences.

10. If two triangles *ABC* and *DEF* are on equal bases *BC* and *EF*, and between the same parallels *AD* and *BF*, any straight line drawn parallel to *AD* will cut off equal areas from *ABC* and *DEF*.

11. The diagonals of a trapezium cut each other in the same ratio.

12. The radius of a circle is a mean proportional between the segments of any tangent made by its point of contact and a pair of parallel tangents.

13. The straight lines which join corresponding angles of two similar triangles whose corresponding sides are parallel will, if produced, meet in a point.

14. The sides of a right-angled triangle are 6 and 8 inches. Find the length of the perpendicular to the hypotenuse.

15. Prove that no straight line can be divided in two points, both external or both internal, in the same ratio.

16. Prove Prop. 6 by the method of superposition.

17. Given the base and the vertical angle of a triangle, find the locus of the point of intersection of the bisectors of the sides.

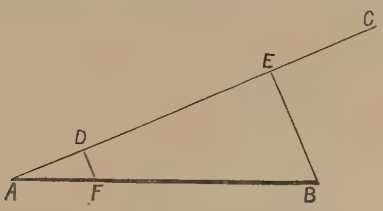
18. *ABC* is a triangle, having the angle *BAC* a right angle, and *CD* and *CE* the internal and external bisectors of the angle at *C*, cutting *AB* in *D* and *E* respectively. Show that *AC* is a mean proportional between *AD* and *AE*.

19. *A*, *P*, *B*, *Q* is a harmonic range (Def. 7), and *AB* is bisected at *O*. Show that *OB* is a mean proportional between *OP* and *OQ*.

20. The rectangle contained by two straight lines is a mean proportional between the squares on the lines.

PROP. 9.—*Problem.*—**From a given finite straight line, to cut off any required sub-multiple.**

Let AB be the given finite straight line;
it is required to cut off from AB a given sub-multiple.



Construction.—From A draw a straight line AC of indefinite length making any angle with AB.

In AC take any point D; and, by cutting off in AC successive parts each equal to AD, make AE the same multiple of AD that AB is required to be of the part to be cut off from AB.

Join BE;

and through D draw DF parallel to BE, meeting AB in F. *I. 31*

Then AF shall be the required sub-multiple of AB.

Proof.—Because DF is parallel to BE, a side of $\triangle ABE$;

$\therefore$ $BF : FA :: ED : DA$; *VI. 2*

$\therefore$ by composition, $BA : FA :: EA : DA$. *V. 18*

But AE is the required multiple of AD;

$\therefore$ AB is the required multiple of AF;

that is, AF is the required sub-multiple of AB.

Q. E. F.

EXERCISES.

1. Explain the terms *multiple, equi-multiple, sub-multiple, common multiple.*

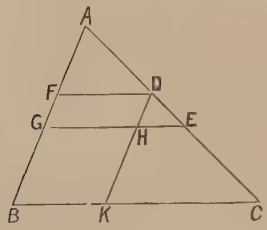
2. In the figure of Prop. 8, show that the ratio $BD : DC$ is the same as the duplicate ratio of $AB : AC$.

3. Given a line $\frac{3}{7}$ of an inch long, draw a line 2 inches long.

4. From a given triangle, show how to cut off one-fifth of its area.

PROP. 10.—*Problem.*—**To divide a given finite straight line similarly to a given divided straight line.**

Let AB be the given finite straight line, and AC the given divided straight line;
it is required to divide AB similarly to AC.

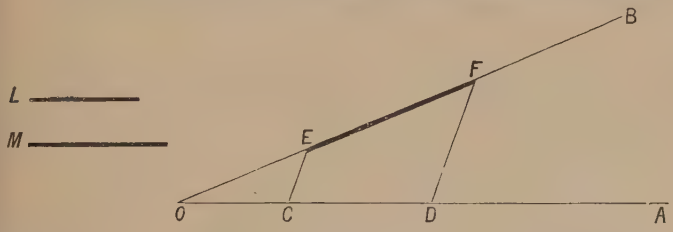


Construction.—Let AC be divided into any parts at the points D and E; and let AB and AC be placed so as to form any angle.
Join BC;
and through D and E draw the straight lines DF and EG parallel to BC, and meeting AB in F and G. *I. 31*
Then AB shall be divided at F and G similarly to AC.
Through D draw DHK parallel to AB.

Proof.—Because each of the figures FH and HB is a parallelogram, *Constr.*
$\therefore$ $DH = FG$ and $HK = GB$. *I. 34*
Because HE is drawn parallel to KC, a side of $\triangle DKC$,
$\therefore$ $DH : HK :: DE : EC$. *VI. 2*
But $DH = FG$ and $HK = GB$;
$\therefore$ $FG : GB :: DE : EC$.
Again, because FD is drawn parallel to GE, a side of $\triangle AGE$,
$\therefore$ $AF : FG :: AD : DE$, *VI. 2*
and $FG : GB :: DE : EC$; *Proved*
$\therefore$ *ex aequali*, $AF : GB :: AD : EC$; *V. 22*
$\therefore$ AB is divided similarly to AC. Q. E. F.

PROP. 11.—*Problem.*—**To find a third proportional to two given finite straight lines.**

Let L and M be the two given finite straight lines;
it is required to find a third proportional to L and M.



Construction.—Draw two straight lines OA and OB of indefinite length from any point O and making any angle AOB;
from OA cut off OC equal to L, and CD equal to M; *I. 3*
from OB cut off OE equal to M or CD;
join CE;
through D draw DF parallel to CE, meeting OB in F. *I. 31*
Then EF shall be a third proportional to L and M.

Proof.—Because CE is drawn parallel to DF, a side of $\triangle ODF$,
$\therefore\ OC : CD :: OE : EF$. *VI. 2*
But $OC = L$, $CD = M$, and $OE = M$; *Constr.*
$\therefore\ L : M :: M : EF$;
therefore EF is a third proportional to L and M.
Q. E. F.

EXERCISES.

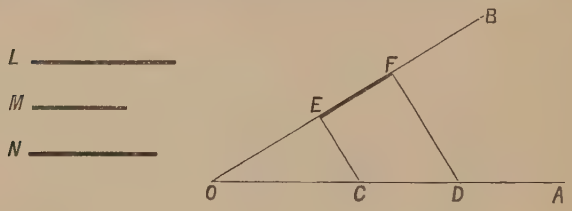
1. A given straight line AB is divided into any two parts AP and PB. Find a point Q in AB produced such that AQ shall be a third proportional to AP and AB.

2. A, B, and C are three points in order along a given straight line. Find a point D in AB such that AD is a third proportional to AC and AB.

3. Show from this Proposition that there is one, and only one, point in a given straight line by which it is divided in a given ratio.

PROP. 12.—*Problem.*—**To find a fourth proportional to three given finite straight lines.**

Let L, M, and N be the three given finite straight lines; it is required to find a fourth proportional to L, M, and N.



Construction.—Draw two straight lines OA and OB of indefinite length from any point O and making any angle AOB;

from OA cut off OC equal to L, and CD equal to M; *I. 3*

from OB cut off OE equal to N;

join CE;

through D draw DF parallel to CE meeting OB in F. *I. 31*

Then EF shall be a fourth proportional to L, M, and N.

Proof.—Because CE is drawn parallel to DF, a side of $\triangle ODF$,

$\therefore\ OC : CD :: OE : EF.$ *VI. 2*

But $OC = L$, $CD = M$, and $OE = N$;

$\therefore\ L : M :: N : EF$;

therefore EF is a fourth proportional to L, M, N.

Q. E. F.

EXERCISES.

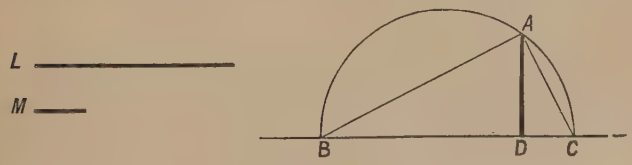
1. A given straight line AB is divided into any three parts AB, BC, and CD. Show how to find a point P in AD such that

$$PA : PB :: PC : PD.$$

2. AOB is a given angle, which is bisected by the line OC. From any point P in OA draw a line PQ to meet OB in Q, such that PQ is divided in a given ratio by OC.

PROP. 13. — *Problem.* — **To find a mean proportional between two given finite straight lines.**

Let L and M be the two given finite straight lines;
it is required to find a mean proportional between L and M.



Construction.—Take any straight line, and in it cut off BD equal to L and DC equal to M;
on BC as diameter describe the semicircle BAC;
from D draw DA at right angles to BC, meeting the circumference in A. *I. 11*
Then DA shall be a mean proportional between L and M.
Join BA and AC.

Proof.—Because BAC is a semicircle,
$\therefore$ $\angle BAC$ is a right angle; *III. 31*
and AD is drawn from the right angle perpendicular to the hypotenuse BC;
$\therefore$ AD is a mean proportional between BD and DC; *VI. 8, Cor. 1*
$\therefore$ $BD : DA :: DA : DC$.
But $BD = L$ and $DC = M$; *Constr.*
$\therefore$ $L : DA :: DA : M$;
$\therefore$ DA is a mean proportional between L and M. Q.E.F.

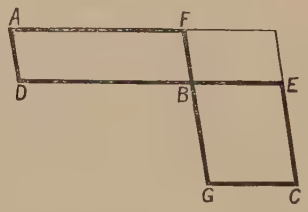
EXERCISE.

In any circle the perpendicular let fall from any point P on any diameter is a mean proportional between the perpendiculars let fall from P on the tangents at the extremities of the diameter.

PROP. 14.—*Theorem.*—(i.) **Parallelograms which are equal in area and which have a pair of equal angles have their sides about the equal angles reciprocally proportional;**

and, conversely, (ii.) **Parallelograms which have a pair of equal angles and which have their sides about the equal angles reciprocally proportional are equal in area.**

(i.) Let AB and BC be two parallelograms which are equal in area and which have the angle FBD equal to the angle GBE; then the sides about these equal angles shall be reciprocally proportional, that is, $DB : BE :: GB : BF$.



Construction.—Place the parallelograms AB and BC so that DB and BE may be in the same straight line;

$\therefore$ $\angle$'s DBF, FBE = two right angles; *I. 13*

$\therefore$ $\angle$'s GBE, FBE = two right angles;

$\therefore$ GB and BF are in the same straight line. *I. 14*

Complete the parallelogram FE. *I. 31*

Proof.—Because $\square AB = \square BC$, *Hyp.*

and FE is another parallelogram,

$\therefore$ $\square AB : \square FE :: \square BC : \square FE$; *V. 7*

but $\square AB : \square FE :: DB : BE$, *VI. 1*

and $\square BC : \square FE :: GB : BF$; *VI. 1*

$\therefore$ $DB : BE :: GB : BF$. *V. 11*

Q. E. D.

(ii.) Conversely, let the angle DBF be equal to the angle GBE, and let $DB : BE :: GB : BF$;

then the parallelogram AB shall be equal in area to the parallelogram BC.

Make the same construction as before.

Proof.—Because $DB : BE :: GB : BF$, *Hyp.*

but $DB : BE :: \square AB : \square FE$ *VI. 1*

and $GB : BF :: \square BC : \square FE$, *VI. 1*

$\therefore \square AB : \square FE :: \square BC : \square FE$; *V. 11*

$\therefore \square AB = \square BC$. *V. 9*

Q. E. D.

EXERCISES.

1. Parallelograms which are equal in area and which have their sides reciprocally proportional are equiangular to one another.

2. The complements of parallelograms about the diagonal of any parallelogram have their sides reciprocally proportional.

3. Describe a rhombus equal to a given rectilineal figure and having an angle equal to a given angle.

4. From a given parallelogram cut off a similar parallelogram by a straight line parallel to two of its sides.

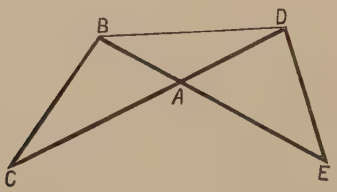
5. Divide a given straight line into two parts which shall have a given ratio. In how many ways can this be done usually?

6. $ABCD$ is a parallelogram and BD its diagonal. From A a straight line $AEFG$ is drawn cutting BD in E, CD in F, and BC produced in G. Prove that AE is a mean proportional between EF and FG.

PROP. 15.—*Theorem.*—(i.) **Triangles which are equal in area and which have a pair of equal angles have their sides about the equal angles reciprocally proportional;**

and, conversely, (ii.) **Triangles which have a pair of equal angles and which have the sides about the equal angles reciprocally proportional are equal in area.**

Let ABC and ADE be two triangles which are equal in area and which have the angle BAC equal to the angle DAE;
then the sides about these equal angles shall be reciprocally proportional;
that is, $CA : AD :: EA : AB$.



Construction.—Place the triangles ABC and ADE so that CA and AD may be in the same straight line;
$\therefore$ $\angle$'s BAC, BAD = two right angles; *I. 13*
$\therefore$ $\angle$'s EAD, BAD = two right angles;
$\therefore$ EA and AB are in the same straight line. *I. 14*
Join BD.

Proof.—Because $\triangle ABC = \triangle ADE$, *Hyp.*
and ABD is another triangle,
$\therefore$ $\triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$; *V. 7*
But $\triangle ABC : \triangle ABD :: CA : AD$, *VI. 1*
and $\triangle ADE : \triangle ABD :: EA : AB$, *VI. 1*
$\therefore$ $CA : AD :: EA : AB$. *V. 11*
Q. E. D.

(ii.) Conversely, let the angle BAC be equal to the angle DAE;
and let $CA : AD :: EA : AB$;
then the triangle ABC shall be equal in area to the triangle ADE.
Make the same construction as before.

Proof.—Because $CA : AD :: EA : AB$ *Hyp.*
and $CA : AD :: \triangle ABC : \triangle ABD$ *VI. 1*
and $EA : AB :: \triangle ADE : \triangle ABD$; *VI. 1*
$\therefore \triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$; *V. 11*
$\therefore \triangle ABC = \triangle ADE$. *V. 9*
Q. E. D.

EXERCISES.

1. Triangles which are equal in area and which have their sides about a pair of angles reciprocally proportional have those angles either equal or supplementary.

2. ABC is a triangle having the side BC produced to D, and CD is equal to BC; AB is bisected in E. Show that the line ED will cut AC in a point P such that $EP : PD :: CP : PA$.

3. In the figure of Prop. 15 prove that the triangles CAE and BAD are similar.

4. Describe an isosceles triangle, having given the area and the vertical angle.

5. Two circles APC and BPD touch each other at P, and straight lines APB and CPD are drawn through P, cutting the circles in A, B, C, and D; join BC and AD. Prove that the triangles APD and BPC are equal in area.

6. In the figure of Prop. 15 prove that CE and BD are parallel.

7. If in the figure of Prop. 15 CB and ED produced meet in F, then FA produced will bisect CE.

8. On AB and AC, two sides of the triangle ABC, squares are described externally, viz., $ABPQ$ and $ACRS$. Prove that the triangle PAS is equal to the triangle QAR in area.

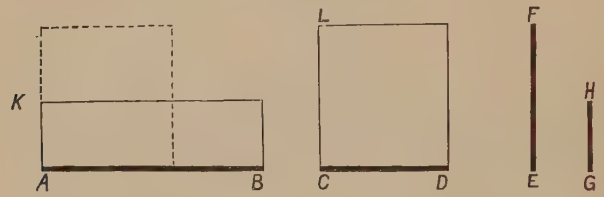
PROP. 16.—*Theorem.*—(i.) **If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means;**

and, conversely, (ii.) **If the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.**

(i.) Let the four straight lines AB, CD, EF, and GH be proportional,

so that $AB : CD :: EF : GH$;

then the rectangle $AB.GH$ shall be equal to the rectangle $CD.EF$.



Construction.—From A draw AK at right angles to AB, and make AK equal to GH;

from C draw CL at right angles to CD, and make CL equal to EF;

complete the parallelograms KB and LD.

Proof.—Because $AB : CD :: EF : GH$, *Hyp.*

and $EF = CL$ and $GH = AK$,

$\therefore$ $AB : CD :: CL : AK$,

and $\angle KAB = \angle LCD$, being right angles;

$\therefore$ the parallelograms KB and LD have a pair of equal angles;

and their sides about the equal angles reciprocally proportional;

$\therefore$ $\square KB = \square LD$. *VI. 14*

But $\square KB$ = the rectangle $AB.GH$, for $AK = GH$, *Constr.*

and $\square LD$ = the rectangle $CD.EF$, for $CL = EF$; *Constr.*

$\therefore$ the rectangle $AB.GH$ = the rectangle $CD.EF$.

Q. E. D.

(ii.) Conversely, let the rectangle $AB.GH$ = the rectangle $CD.EF$; then shall $AB : CD :: EF : GH$.

Make the same construction as before.

Proof.—Because the rectangle $AB.GH$ = the rectangle $CD.EF$, *Hyp.*
and the rectangle $AB.GH = \square KB$, for $GH = AK$, *Constr.*
and the rectangle $CD.EF = \square LD$, for $EF = CL$, *Constr.*
$\therefore \square KB = \square LD$,
and $\angle KAB = \angle LCD$, being right angles;
$\therefore$ the parallelograms KB and LD have a pair of equal angles, and are equal in area;
$\therefore$ their sides about the equal angles are reciprocally proportional; *VI. 14*
$\therefore AB : CD :: CL : AK$;
$\therefore AB : CD :: EF : GH$. Q. E. D.

EXERCISES.

1. Employ Prop. 16 to prove that, if two chords of a circle intersect, either within or without the circle, then the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

2. If a, b, c represent the lengths of three straight lines, show how to draw a line whose length is represented by $\frac{ab}{c}$.

3. If a, b, c represent the lengths of the sides of a right-angled triangle where c is the hypothenuse, and if p is the length of the perpendicular on the hypothenuse, prove that—

(i.) $ab = cp$; (ii.) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$.

4. A triangle ABC is inscribed in a circle, and BD is drawn parallel to the tangent at A to meet AC, produced if necessary, in D. Show that the rectangle $AB.BC$ is equal to the rectangle $AC.BD$.

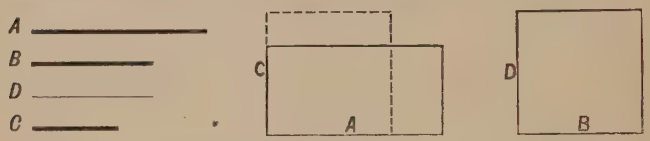
5. On a given finite straight line AB describe an isosceles triangle equal in area to a given triangle.

6. ABC is a triangle inscribed in a circle, and the exterior angle at A is bisected by a straight line which meets the circle in E and BC in D. Show that the rectangle $AB.AC$ is equal to the rectangle $EA.AD$.

PROP. 17.—*Theorem.*—(i.) **If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean;**

and, conversely, (ii.) **If the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportional;**

(i.) Let the three straight lines A, B, and C be proportional,
so that $A : B :: B : C$;
then the rectangle $A.C$ shall be equal to the square on B.



Construction.—Draw another straight line D equal to B.

Proof.—Because $A : B :: B : C$, *Hyp.*
$\therefore$ $A : B :: D : C$, for $D = B$;
$\therefore$ the rectangle $A.C$ = the rectangle $B.D$. *VI. 16*
But the rectangle $B.D$ = the square on B, for $D = B$;
$\therefore$ the rectangle $A.C$ = the square on B. Q. E. D.

(ii.) Conversely, let the rectangle $A.C$ = the square on B;
then shall $A : B :: B : C$.

Make the same construction as before.

Proof.—Because the rectangle $A.C$ = the square on B, *Hyp.*
and the square on B = the rectangle $B.D$, for $D = B$;
$\therefore$ the rectangle $A.C$ = the rectangle $B.D$;
$\therefore$ $A : B :: D : C$;
$\therefore$ $A : B :: B : C$. Q. E. D.

EXERCISE.

If a, b, c, d represent the lengths of four straight lines, show how to construct the lengths represented by (i.) $\frac{b^2}{a}$, (ii.) $\sqrt{ab}$, (iii.) $\frac{a^2+b^2}{c}$.

Notes on Propositions 6 to 17.

Propositions 6 and 7.—Like Prop. 5, Props. 6 and 7 both depend on Prop. 4. These four propositions all deal with similar triangles. Euclid gives Prop. 7 as it is stated in the Corollary to Prop. 7.

Proposition 8.—This is really an example of Prop. 4, that "triangles which are equiangular to one another are similar." It is treated specially because it is a most important theorem. The three Corollaries are also important, and are often used in riders.

Propositions 9, 10, 11, 12, 13.—Euclid uses the word *part* in Prop. 9 instead of sub-multiple, but he takes *part* to mean the same as sub-multiple. Props. 9, 10, 11, 12 are all proved by VI. 2; and Prop. 13 depends on VI. 8.

A most important application of Prop. 10 is to divide a straight line internally and externally in a given ratio. (See Def. 7, and compare Props. 3 and A.)

Prop. 11 is a special case of Prop. 12; if the lines M and N are made equal in Prop. 12, then Prop. 12 is the same as Prop. 11.

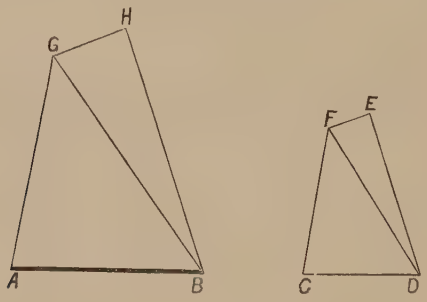
Propositions 14 and 15.—Either of these propositions might be inferred from the other, and the two propositions might have been included in one proposition, just as triangles and parallelograms are both dealt with in VI. 1. The area of a parallelogram is double the area of a triangle with the same base and the same altitude (I. 41). Several riders on these propositions may be solved by looking upon the parallelograms AB and BC in Prop. 14 as the complements of the parallelograms FE and DG in the parallelogram AC, and by noticing that the triangles BCD and BED in Prop. 15 are equal in area. The proof of both propositions depends on VI. 1.

Propositions 16 and 17.—Prop. 16 is a special case of Prop. 14 (ii.), the equal angles being right angles; and Prop. 17 is a special case of Prop. 16, the line CD being equal to EF. These propositions are most important in Algebra and Arithmetic. In Algebra many theorems in Proportion are proved by assuming that the product of the extremes is equal to the product of the means; and in Arithmetic all rules that are worked by Proportion depend on the same theorem. (See *Tutorial Arithmetic*, § 177.)

PROP. 18.—*Problem.*—**On a given straight line, to describe a rectilineal figure similar and similarly described to a given rectilineal figure.**

Let *AB* be the given straight line, and *CDEF* the given figure of four sides;

it is required to describe on *AB* a rectilineal figure similar and similarly described to *CDEF*.



Construction.—Join *DF*.

At the point *A* in the straight line *AB* make the angle *BAG* equal to the angle *DCF*; *I. 23*

and at the point *B* in the straight line *AB* make the angle *ABG* equal to the angle *CDF*; *I. 23*

$\therefore \quad \angle AGB = \angle CFD$; *I. 32*

$\therefore$ the triangle *AGB* is equiangular and therefore similar to the triangle *CFD*. *VI. 4*

Again, at the point *B* in the straight line *BG* make the angle *GBH* equal to the angle *FDE*;

and at the point *G* in the straight line *BG* make the angle *BGH* equal to the angle *DFE*;

$\therefore \quad \angle GHB = \angle FED$; *I. 32*

$\therefore$ the triangle *GHB* is equiangular and therefore similar to the triangle *FED*. *VI. 4*

Then *ABHG* shall be the figure required.

Proof.—(1) Because $\angle AGB = \angle CFD$, *Constr.*
and $\angle BGH = \angle DFE$, *Constr.*
$\therefore$ the whole $\angle AGH =$ the whole $\angle CFE$.

Similarly the whole $\angle ABH$ can be proved equal to the whole $\angle CDE$; and the $\angle$'s at A and at H are equal respectively to the $\angle$'s at C and at E;

$\therefore$ the figure $ABHG$ is equiangular to the figure $CDEF$.

(2) Again, because $\triangle AGB$ is similar to $\triangle CFD$,

$\therefore$ $AG : GB :: CF : FD$;

and because $\triangle GHB$ is similar to $\triangle FED$,

$\therefore$ $GB : GH :: FD : FE$;

$\therefore$ *ex aequali*, $AG : GH :: CF : FE$. *V. 22*

Similarly, it can be proved that $AB : BH :: CD : DE$.

And, from the similar triangles it follows that

$BA : AG :: DC : CF$,

and that $GH : HB :: FE : ED$;

$\therefore$ the figures $ABHG$ and $CDEF$ are equiangular, and have their sides about the equal angles proportional;

$\therefore$ $ABHG$ is similar to $CDEF$; *VI. Def. 2*

and AB and CD are corresponding sides of the figures;

$\therefore$ $ABHG$ is similarly described to $CDEF$. Q. E. F.

In the same way a rectilineal figure may be described similar and similarly described to a given rectilineal figure of five or of any number of sides; namely, by dividing the given figure into triangles by straight lines drawn from the same vertex, and so on.

EXERCISES.

1. Describe a triangle similar to a given triangle and having a given perimeter.

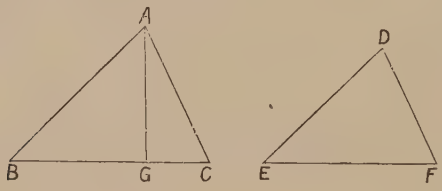
2. Show how to inscribe in a given triangle another triangle whose sides shall be parallel to three given straight lines.

3. In a given triangle inscribe a square.

PROP. 19.—*Theorem.*—**Similar triangles are to one another in the duplicate ratio of their corresponding sides.**

Let ABC and DEF be similar triangles, and let BC and EF be corresponding sides;

then shall $\triangle ABC$ be to $\triangle DEF$ in the duplicate ratio of BC to EF.



Construction.—From BC cut off BG, a third proportional to BC and EF, so that $BC : EF :: EF : BG$. Join AG. *VI. 11*

Proof.—Because $\triangle$'s ABC and DEF are similar,

$\therefore$ $AB : BC :: DE : EF$;

$\therefore$ alternately, $AB : DE :: BC : EF$. *V. 16*

But $BC : EF :: EF : BG$;

$\therefore$ $AB : DE :: EF : BG$ *V. 11*

and $\angle ABC = \angle DEF$;

$\therefore$ the two $\triangle$'s ABG and DEF have a pair of equal angles, and their sides about the equal angles reciprocally proportional;

$\therefore$ $\triangle ABG = \triangle DEF$, in area. *VI. 15*

Again, because $BC : EF :: EF : BG$;

$\therefore$ $BC : BG$ in the duplicate ratio of BC to EF. *V. Def. 10*

But $\triangle ABC : \triangle ABG :: BC : BG$; *VI. 1*

$\therefore$ $\triangle ABC : \triangle ABG$ in the duplicate ratio of BC to EF,

and $\triangle ABG = \triangle DEF$; *Proved*

$\therefore$ $\triangle ABC : \triangle DEF$ in the duplicate ratio of BC to EF. Q. E. D.

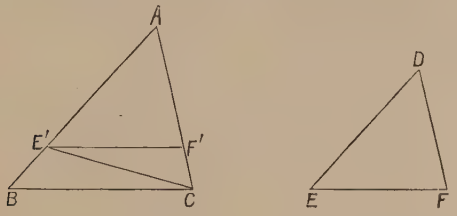
Corollary.—If three straight lines be proportionals, as the first is to the third, so is any triangle described on the first to a similar and similarly described triangle on the second.

Alternative Proof.

PROP. 19.—*Theorem.*—**Similar triangles are to one another in the duplicate ratio of their corresponding sides.**

Let ABC and DEF be similar triangles, and let AB and DE be corresponding sides;

then shall $\triangle ABC$ be to $\triangle DEF$ in the duplicate ratio of BC to EF.



Construction.—Place $\triangle DEF$ on $\triangle ABC$, so that the point D coincides with the point A, and DE falls along AB;

then, because $\angle EDF = \angle BAC$,

$\therefore$ DF will fall along AC.

Let E' and F' represent the new positions of E and F, so that $AE' = DE$ and $AF' = DF$. Join $E'C$.

Proof.—Because $\triangle ABC$ is similar to $\triangle DEF$, *Hyp.*

$\therefore$ $AB : DE :: AC : DF$; *VI. Def. 2*

but $AE' = DE$ and $AF' = DF$;

$\therefore$ $AB : AE' :: AC : AF'$.

Again, because $\triangle ABC : \triangle AE'C :: AB : AE'$ *VI. 1*

and $\triangle AE'C : \triangle AE'F' :: AC : AF'$, *VI. 1*

$\therefore$ $\triangle ABC : \triangle AE'C :: \triangle AE'C : \triangle AE'F'$; [*V. Def. 10*

$\therefore$ $\triangle ABC : \triangle AE'F'$ in the duplicate ratio of $\triangle ABC$ to $\triangle AE'C$; that is, in the duplicate ratio of AB to AE'; that is, of AB to DE.

Q. E. D.

Corollary.—If three straight lines be proportionals, as the first is to the third, so is any triangle described on the first to a similar and similarly described triangle on the second.

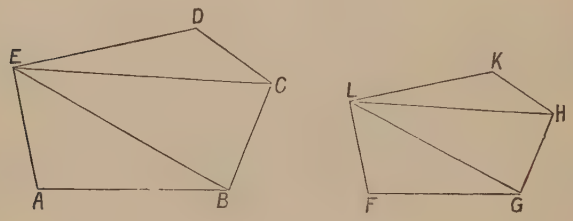
PROP. 20.—*Theorem.*—**Similar polygons (i.) may be divided into the same number of similar triangles; (ii.) having each to each the same ratio that the polygons have; and (iii.) the polygons are to each other in the duplicate ratio of their corresponding sides.**

Let *ABCDE* and *FGHKL* be similar polygons, and let *AB* and *FG* be corresponding sides;

then (i.) the polygons *ABCDE* and *FGHKL* may be divided into the same number of similar triangles;

(ii.) these triangles shall have each to each the same ratio that the polygons have; and

(iii.) the polygon *ABCDE* shall be to the polygon *FGHKL* in the duplicate ratio of *AB* to *FG*,



Construction.—Join *EB*, *EC*, *LG*, *LH*.

Proof.—(i.) Because the polygon *ABCDE* is similar to the polygon *FGHKL*; *Hyp.*

$\therefore \angle EAB = \angle LFG$;

and $EA : AB :: LF : FG$; *VI. Def. 2*

$\therefore$ $\triangle EAB$ is similar to $\triangle LFG$; *VI. 6*

$\therefore \angle ABE = \angle FGL$.

But, because the polygons are similar, *Hyp.*

$\therefore$ the whole $\angle ABC$ = the whole $\angle FGH$;

$\therefore$ the remainder, $\angle EBC$ = the remainder, $\angle LGH$.

And, because $\triangle EAB$ is similar to $\triangle LFG$, *Proved*

$\therefore EB : AB :: LG : FG$;

and, because the polygons are similar,

$\therefore$ $AB : BC :: FG : GH$;

$\therefore$ *ex aequali*, $EB : BC :: LG : GH$, *V. 22*

that is, the sides about the equal angles EBC and LGH are proportionals;

$\therefore$ $\triangle EBC$ is similar to $\triangle LGH$. *VI. 6*

Similarly it may be proved that $\triangle ECD$ is similar to $\triangle LHK$;

$\therefore$ the polygons $ABCDE$ and $FGHKL$ have been divided into the same number of similar triangles. Q. E. D.

(ii.) Again, because $\triangle ABE$ is similar to $\triangle FGL$,

$\therefore$ $\triangle ABE$ is to $\triangle FGL$ in the duplicate ratio of EB to LG; *VI. 19*

similarly, $\triangle EBC$ is to $\triangle LGH$ in the duplicate ratio of EB to LG;

$\therefore$ $\triangle ABE : \triangle FGL :: \triangle EBC : \triangle LGH$. *V. 11*

Similarly it can be proved that

$\triangle EBC : \triangle LGH :: \triangle ECD : \triangle LHK$;

$\therefore$ $\triangle ABE : \triangle FGL :: \triangle EBC : \triangle LGH$,

$:: \triangle ECD : \triangle LHK$;

$\therefore$ $\triangle ABE : \triangle FGL$ :: sum of antecedents : sum of consequents; *V. 12*

that is, $\triangle ABE : \triangle FGL$ :: polygon $ABCDE$: polygon $FGHKL$. Q. E. D.

(iii.) Again, because

$\triangle ABE : \triangle FGL$ :: the polygon $ABCDE$: the polygon $FGHKL$,

and, $\triangle ABE : \triangle FGL$ in the duplicate ratio of AB to FG, *VI. 19*

$\therefore$ the polygon $ABCDE$: the polygon $FGHKL$ in the duplicate ratio of AB to FG. Q. E. D.

Corollary 1.—If three straight lines be proportional, then, as the first is to the third, so is any rectilineal figure described on the first to the similar and similarly described rectilineal figure on the second. For, let X be a third proportional to AB and FG,

so that AB is to X in the duplicate ratio of AB to FG;

but $ABCDE$ is to $FGHKL$ in the duplicate ratio of AB to FG;

$\therefore$ $AB : X :: ABCDE : FGHKL$.

Corollary 2.—The ratio of two squares to each other is the same as the duplicate ratio of the lines on which the squares are described.

For the squares are similar figures, and the lines on which they are described are corresponding sides.

Corollary 3.—Similar and similarly described figures are to each other as the squares described on their corresponding sides. (By Cor. 2.)

Corollary 4.—Similar and similarly described figures that are equal in area have their corresponding sides equal. (By Cor. 3).

Notes on Propositions 18, 19, 20.

Proposition 18.—Euclid solves this problem for a five-sided figure; but the method of construction and proof is exactly similar to that for a quadrilateral. It is also evident that the same method is applicable to a triangle, or to a figure of any number of sides. Euclid also uses the words "similarly situated" instead of "similarly described"; but modern geometricians take "similarly situated" to mean figures whose sides are parallel.

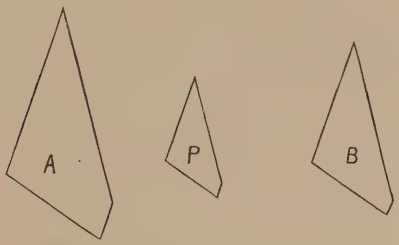
Proposition 19.—Euclid's enunciation of this proposition is as follows:—"Similar triangles are to one another in the duplicate ratio of their homologous sides." The word "corresponding" is used instead of "homologous" because it is easier to understand. Beginners usually find this proposition difficult, and the difficulty arises from not understanding duplicate ratio. The student should therefore read what is said about duplicate ratio in Book V. before reading VI. 19.

The first proof is Euclid's. The proposition is also remarkable for the many references it makes to other propositions in Books V. and VI. The beginner therefore will find it advisable to revise his work and to go over Props. 1–18 again before attempting Prop. 19. The Corollary is important, but it is included in Corollary 1 to Prop. 20.

Proposition 20.—Corollary 2 shows that Euclid's definition of duplicate ratio is really the same as the algebraical definition of duplicate ratio.

PROP. 21.—*Theorem.*—**Rectilinear figures which are similar to the same rectilinear figure are similar to one another.**

Let *A*, *B*, and *P* be three rectilineal figures, of which *A* and *B* are each similar to *P*;
then *A* shall be similar to *B*.



Proof.—Because *A* is similar to *P*,
∴ *A* is equiangular to *P*, and the sides about their equal angles are proportional. *VI. Def. 2*
Again, because *B* is similar to *P*,
∴ *B* is equiangular to *P*, and the sides about their equal angles are proportional. *VI. Def. 2*
∴ *A* and *B* are each of them equiangular to *P*, and have their sides about the equal angles proportional to the corresponding sides of *P*;
∴ *A* is equiangular to *B*, and the sides about their equal angles are proportional; *V. 11*
∴ *A* is similar to *B*. *VI. Def. 2*
Q. E. D.

EXERCISES.

1. If *ABCD* and *PQRS* are two equiangular quadrilaterals, such that
$$AB : PQ :: BC : QR,$$
prove that the two quadrilaterals are similar.

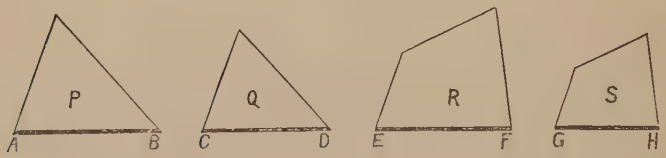
2. On a given straight line describe a quadrilateral similar but oppositely situated to a given quadrilateral.

PROP. 22.—*Theorem.*—**If a pair of similar rectilineal figures be described on two given straight lines, and if another pair of similar rectilineal figures be described on two other straight lines, then (i.) if the four lines be proportional, the figures shall be proportional; *and, conversely,* (ii.) if the figures be proportional, the lines shall be proportional.**

Let P and Q be a pair of similar rectilineal figures described on the two given straight lines AB and CD; and let R and S be another pair of similar rectilineal figures described on the two given straight lines EF and GH;

then (i.), if $AB : CD :: EF : GH$,

then shall $P : Q :: R : S$.



Proof.—Because $AB : CD :: EF : GH$,

$\therefore$ the duplicate ratio of AB to CD is the same as the duplicate ratio of EF to GH; *V., Add. Prop. 1*

but $P : Q$ in the duplicate ratio of AB to CD *VI. 20*

and $R : S$ in the duplicate ratio of EF to GH,

$\therefore$ $P : Q :: R : S$. Q. E. D.

(ii.) Conversely, if $P : Q :: R : S$,

then shall $AB : CD :: EF : GH$.

Proof.—Because $P : Q :: R : S$,

and $P : Q$ in the duplicate ratio of AB to CD, *VI. 20*

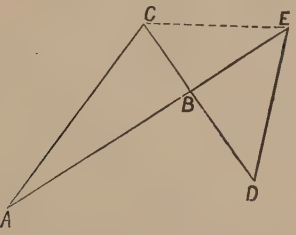
and $R : S$ in the duplicate ratio of EF to GH; *VI. 20*

$\therefore$ the duplicate ratio of AB to CD is the same as the duplicate ratio of EF to GH;

$\therefore$ $AB : CD :: EF : GH$. *V., Add. Prop. 2.* Q. E. D.

PROP. 23.—*Theorem.*—**If two triangles have one angle of the one equal to one angle of the other, then the areas of the triangles shall have the same ratio to one another as the ratio which is compounded of the ratios of the sides about the equal angles.**

Let *ABC* and *BDE* be two triangles, having $\angle ABC$ equal to $\angle DBE$; then the areas of $\triangle$'s *ABC* and *BDE* shall have to one another the same ratio as the ratio compounded of the ratios of *AB* to *BE* and *CB* to *BD*.



Construction.—Let the two triangles *ABC* and *BDE* be placed so that *AB* and *BE* are in the same straight line;

$\therefore$ *CB* and *BD* are in the same straight line; *I. 14*

join *CE*.

Proof.—Because $\triangle ABC : \triangle CBE :: AB : BE$, *VI. 1*

and $\triangle CBE : \triangle DBE :: CB : BD$, *VI. 1*

$\therefore$ $\triangle ABC$ has to $\triangle DBE$ the same ratio as the ratio compounded of the ratios *AB* to *BE* and *CB* to *BD*. *V., Def. 11*

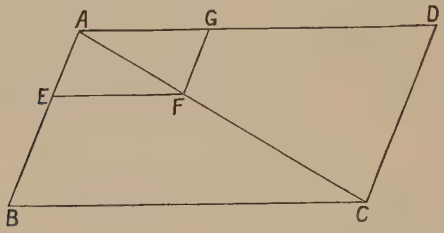
Q. E. D.

Corollary.—Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratio of their sides.

For two parallelograms which have one angle of the one equal to one angle of the other have all their angles equal each to each, and are therefore equiangular to one another; and every parallelogram is bisected by each of its diagonals.

PROP. 24. — *Theorem.* — **Any parallelogram about the diagonal of another parallelogram is similar to it.**

Let *ABCD* be a parallelogram and *AC* its diagonal, and let *AEFG* be any parallelogram about the diagonal *AC*; then the parallelogram *AEFG* shall be similar to the parallelogram *ABCD*.



Proof.— (i.) Because *EF* is parallel to *BC*,

$\therefore \quad \angle AEF = \angle ABC$; *I. 29*

and $\angle EAG$ is the same as $\angle BAD$;

and the opposite angles of parallelograms are equal; *I. 34*

$\therefore$ the parallelogram *AEFG* is equiangular to the parallelogram *ABCD*.

(ii.) And, because $\triangle AEF$ is equiangular and therefore similar to $\triangle ABC$, *VI. 4*

$\therefore \quad AE : EF :: AB : BC$,

and the opposite sides of parallelograms are equal; *I. 34*

$\therefore$ the ratio of any pair of sides of the parallelogram *AEFG* is the same as the ratio of the corresponding sides of the parallelogram *ABCD*;

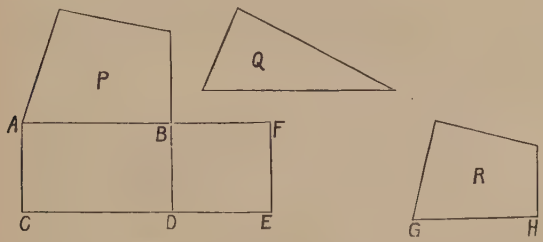
$\therefore$ the parallelograms *AEFG* and *ABCD* are equiangular and have the sides about their equal angles proportional;

$\therefore$ the parallelogram *AEFG* is similar to the parallelogram *ABCD*.

Q. E. D.

PROP. 25.—*Problem.*—**To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another given rectilineal figure.**

Let P and Q be two given rectilineal figures;
it is required to describe a rectilineal figure which shall be similar to P and equal to Q.

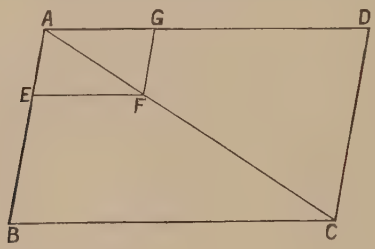


Construction.—On AB, a side of the figure P, describe the rectangle $ACDB$, equal to P; *I. 44, 45*
on BD describe a rectangle $BDEF$ equal to Q.
Find GH, a mean proportional between AB and BF; *VI. 13*
and on GH describe the rectilineal figure R, similar and similarly described to the given figure P. *VI. 18*
Then R shall be the figure required.

Proof.—Because $AB : GH :: GH : BF$, *Constr.*
$\therefore$ $AB : BF$ in the duplicate ratio of AB to GH. *V. Def. 10*
But $AD : BE :: AB : BF$; *VI. 1*
$\therefore$ $AD : BE$ in the duplicate ratio of AB to GH.
But, since P and R are similar figures,
$\therefore$ $P : R$ in the duplicate ratio of AB to GH; *VI. 20.*
$\therefore$ $P : R :: AD : BE$. *V. 11*
But $P = AD$; $\therefore$ $R = BE$. *Constr.*
But BE was made equal to Q; $\therefore$ $R = Q$; *Constr.*
and R is similar to P; *Constr.*
$\therefore$ the figure R has been described similar to the given figure P, and equal to the given figure Q. Q. E. F.

PROP. 26.—*Theorem.*—**If two similar and similarly situated parallelograms have a common angle, they are about the same diagonal.**

Let $ABCD$ and $AEFG$ be two similar and similarly situated parallelograms, and let them have the common angle BAD; then they shall be about the same diagonal.



Construction.—Draw AF and AC.

Proof.—Because the parallelograms are similar,

$\therefore$ $AE : EF :: AB : BC$, *VI. Def. 2*

and $\angle AEF = \angle ABC$, *I. 29*

$\therefore$ $\triangle$'s AEF and ABC are equiangular; *VI. 6*

$\therefore$ $\angle EAF = \angle BAC$;

$\therefore$ AF and AC lie along the same straight line. Q. E. D.

MISCELLANEOUS EXERCISES ON PROPOSITIONS 9–26.

1. If $A : B :: C : D$, prove that the duplicate ratio of A to B is the same as the duplicate ratio of C to D.

2. State and prove the converse of Ex. 1.

3. Give the algebraical definition of *duplicate ratio*, and show that it agrees with Euclid's definition.

4. Show how to describe a triangle similar to a given triangle but having double its area.

5. Similar triangles are to one another as the squares described on their homologous sides.

6. Show how to describe a triangle similar to each of five similar triangles and equal in area to their sum.

7. Bisect a triangle by a straight line parallel to one of its sides.

8. ABC and DEF are two triangles having the angle BAC equal to the angle EDF, and the areas of ABC and DEF are in the duplicate ratio of BC to EF. Show that ABC is similar to DEF.

9. A trapezium $ABCD$ has its sides AB and CD parallel, and AB is three times CD. If the diagonals intersect in O, show that the triangle AOB is nine times the triangle COD.

10. The areas of two triangles are to one another in the ratio compounded of the ratios of their altitudes and of their bases.

11. Similar polygons which have two of their homologous sides equal are equal in all respects.

12. ABC is any triangle, and BE and CF are perpendiculars to AC and AB. Prove that (i.) $\triangle ABE : \triangle ACF ::$ sq. on AB : sq. on AC;

(ii.) $\triangle AEF : \triangle ABC ::$ sq. on EF : sq. on AB.

13. AOC and BOD are two chords of a circle which intersect in O. Prove that $\triangle AOB : \triangle COD ::$ sq. on OA : sq. on OD.

14. Rectangles are to one another in the ratio compounded of the ratio of their sides.

15. Describe an equilateral triangle equal to a given square.

16. Describe a rectangle equal to a given square and having its sides in a given ratio.

17. Inscribe in a given triangle a triangle which shall be similar to another given triangle and have a given area.

18. The diagonals AC and BD of a cyclic quadrilateral intersect in O. Prove that $AO : OC ::$ the rectangle $AB.AD$: the rectangle $BC.CD$.

19. ABC is a right-angled triangle, and CD is drawn perpendicular to the hypotenuse BC. Prove that

$$AD : DB ::$$ sq. on AC : sq. on CB.

20. Find the greatest triangle which can be inscribed in a given triangle so as to have one side parallel to one of the sides of the given triangle.

21. About a given triangle show how to describe the least possible triangle.

22. Similar and similarly described rectilineal figures which are equal in area have their homologous sides equal.

23. $ABCD$ and $EFGH$ are two equiangular quadrilaterals, and

$$AB : EF :: CD : GH.$$

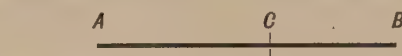
Prove that the quadrilaterals are similar.

24. Is there any exception to the truth of Ex. 23?

25. In a given triangle inscribe a parallelogram similar to a given parallelogram.

PROP. 30.—*Problem.*—**To divide a given straight line in extreme and mean ratio.**

Let AB be the given straight line;
it is required to divide AB in extreme and mean ratio.

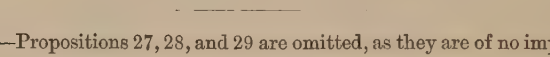


Construction.—Divide AB at the point C into two parts, so that the rectangle $AB.CB$ = the square on AC; *II. 11*
then C shall be the point required.

Proof.—Because the rectangle $AB.CB$ = the square on AC, *Constr.*
$\therefore$ $AB : AC :: AC : CB$. *VI. 17* Q. E. F.

NOTE.—Propositions 27, 28, and 29 are omitted, as they are of no importance.

In the Notes on II. 11 (pages 145, 146) it was stated that the parts AH and HB are incommensurable; but the theorem No. 11, on page 148, will enable us to express the ratio of AH to HB in II. 11, or of AC to CB in VI. 30, in a convenient algebraical form.



Let AB be a straight line divided in extreme and mean ratio, or in medial section, at H.

From AH cut off AK equal to BH; then AH is divided in medial section.
Again, cut off BL equal to HK; then BH is divided in medial section.

$$\frac{AH}{HB} = \frac{AK+KH}{HB} = 1 + \frac{KH}{KB} = 1 + \frac{1}{\frac{HB}{KH}}.$$

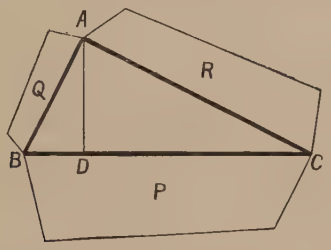
$$\text{Again, } \frac{HB}{KH} = 1 + \frac{HL}{KH} = 1 + \frac{1}{\frac{KH}{HL}}, \text{ and so on;}$$

$$\therefore \frac{AH}{HB} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}, \text{ \&c.}$$

The successive convergents of this continued fraction will approach nearer and nearer to the value of the ratio $\frac{AH}{HB}$; so that, although the ratio $\frac{AH}{HB}$ is incommensurable, we can get a value of $\frac{AH}{HB}$ to any assigned degree of accuracy.

PROP. 31.—*Theorem.*—**In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described rectilineal figures on the sides containing the right angle.**

Let ABC be a right-angled triangle, having the angle BAC a right angle;
and on BC, BA, and AC let the similar and similarly described rectilineal figures P, Q, and R be described;
then the figure P shall be equal to the sum of the figures Q and R.



Construction.—Draw AD perpendicular to BC.

Proof.—Because $BC : AB :: AB : BD$, *VI. 8, Cor. 2*
$\therefore$ $BC : BD :: P : Q$, *VI. 20, Cor. 1*
$\therefore$ inversely, $BD : BC :: Q : P$. *V. B*

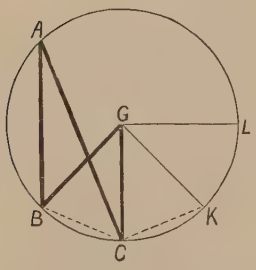
In the same manner it can be proved that
$DC : BC :: R : P$;
$\therefore$ the sum of BD and $DC : BC ::$ the sum of Q and $R : P$. *V. 18*
But the sum of BD and $DC = BC$;
$\therefore$ the sum of Q and $R = P$. Q. E. D.

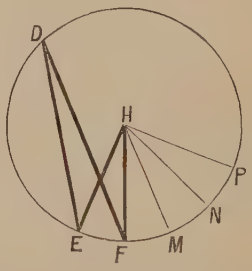
NOTE.—Proposition 32 is omitted because it is defective, and is never used. Its enunciation is as follows:—If two triangles which have two sides of the one proportional to two sides of the other be joined at one angle so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

PROP. 33. — *Theorem.* — **In equal circles, angles, whether at the centres or the circumferences, have the same ratio as the arcs on which they stand; so also have the sectors.**

Let *ABC* and *DEF* be two equal circles, and let *BGC* and *EHF* be angles at the centres, and *BAC* and *EDF* be angles at the circumferences;

then shall (i.) $\angle BGC : \angle EHF ::$ the arc *BC* : the arc *EF*,
and (ii.) $\angle BAC : \angle EDF ::$ the arc *BC* : the arc *EF*,
and (iii.) the sector *BGC* : the sector *EHF*
:: the arc *BC* : the arc *EF*.





Construction.—In the circle *ABC* take any number of arcs *CK*, *KL* each equal to *BC*;
and in the circle *DEF* take any number of arcs *FM*, *MN*, *NP* each equal to *EF*;
join *GK*, *GL*, *HM*, *HN*, *HP*.

Proof.—Because the arcs *BC*, *CK*, *KL* are all equal, *Constr.*
$\therefore$ $\angle$'s *BGC*, *CGK*, *KGL* are all equal; *III. 27, Cor.*
$\therefore$ $\angle$ *BGL* is the same multiple of $\angle$ *BGC* that the arc *BL* is of the arc *BC*.
Similarly, $\angle$ *EHP* is the same multiple of $\angle$ *EHF* that the arc *EP* is of the arc *EF*.
And, if the arc *BL* is greater than, equal to, or less than the arc *EP*,

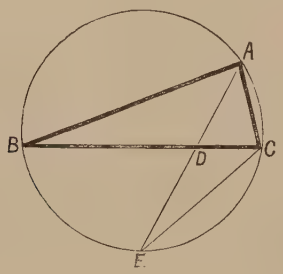
then $\angle BGL$ is greater than, equal to, or less than $\angle EHP$;
$\therefore$ $\angle BGC : \angle EHF ::$ the arc BC : the arc EF. *V. Def. 4* Q. E. D.

(ii.) Again, because $\angle BGC =$ twice $\angle BAC$, *III. 20*
and $\angle EHF =$ twice $\angle EDF$, *III. 20*
$\therefore$ $\angle BAC : \angle EDF ::$ the arc BC : the arc EF. *V. 15* Q. E. D.

(iii.) Join BC, CK.
Because the arc $BC =$ the arc CK, *Constr.*
$\therefore$ the chord $BC =$ the chord CK, *III. 29, Cor.*
and the angles in the smaller segments on BC and CK will be equal; *III. 27, Cor.*
$\therefore$ the smaller segments on BC and CK are similar and equal.
Again, in the two $\triangle$'s BGC, CGK, [*III. 24*
because $BG = GK$, *Radii*
and CG is common to both triangles,
and $\angle BGC = \angle CGK$, being on equal arcs;
$\therefore$ by I. 4, $\triangle BGC = \triangle CGK$ in area. [*III. 27, Cor.*
But smaller segment on $BC =$ smaller segment on CK;
$\therefore$ the whole, the sector $BGC =$ the whole, the sector CGK.
Similarly, it can be proved that
the sector $CGK =$ the sector KGL;
$\therefore$ the sector BGL is the same multiple of the sector BGC that the arc BL is of the arc BC.
Similarly, the sector EHP is the same multiple of the sector EHF that the arc EP is of the arc EF.
And, if the arc BL is greater than, equal to, or less than the arc EP,
then the sector BGL is greater than, equal to, or less than the sector EHP;
$\therefore$ the sector BGC : the sector $EHF ::$ the arc BC : the arc EF. *V. Def. 4*
Q. E. D.

PROP. B.—*Theorem.*—**If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the segments of the base, together with the square on the line which bisects the vertical angle.**

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD, which cuts the base BC in D;
then the rectangle $BA.AC$ shall be equal to the rectangle $BD.DC$, together with the square on AD.



Construction.—Describe a circle about $\triangle ABC$, and produce AD to meet the circumference in E; join EC.

Proof.—Because, in the two $\triangle$'s BAD, EAC,

$\angle BAD = \angle EAC$, *Constr.*

and $\angle ABD = \angle AEC$, being in the same segment, *III. 21*

$\therefore$ $\triangle BAD$ is equiangular, and therefore similar to $\triangle EAC$;

$\therefore$ $BA : DA :: EA : AC$; *VI. 4*

$\therefore$ the rectangle $BA.AC$ = the rectangle $EA.DA$ *VI. 16*

= the rectangle $ED.DA$, together with the square on AD *II. 3*

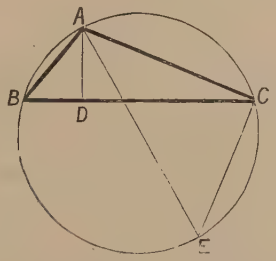
= the rectangle $BD.DC$, together with the square on AD. *III. 35*

Q. E. D.

PROP. C.—*Theorem.*—**If from the vertical angle of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by this perpendicular and the diameter of the circle described about the triangle.**

Let ABC be a triangle, and let AD be drawn perpendicular to the base BC;

then the rectangle $BA.AC$ shall be equal to the rectangle contained by AD and the diameter of the circle described about ABC.



Construction.—Describe a circle about $\triangle ABC$; draw the diameter AE, and join EC.

Proof.—Because, in the two $\triangle$'s BAD, EAC,

the right $\angle BDA$ = the right $\angle ACE$ in the semicircle ACE, *III. 31*

and $\angle ABD = \angle AEC$, in the same segment, *III. 21*

$\therefore$ $\triangle BAD$ is equiangular and therefore similar to $\triangle AEC$;

$\therefore$ $BA : AD :: AE : AC$; *VI. 4*

$\therefore$ the rectangle $BA.AC$ = the rectangle $AD.AC$. *VI. 16*

Q. E. D.

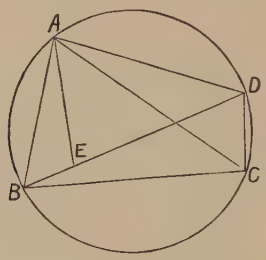
EXERCISES.

1. State and prove the converse of Proposition C.

2. In an isosceles triangle each of the equal sides is a mean proportional between the perpendicular drawn from the vertical angle to the base and the diameter of the circle described about the triangle.

PROP. D.—*Ptolemy's Theorem.*—**If a quadrilateral be inscribed in a circle, the rectangle contained by its diagonals shall be equal to the sum of the two rectangles contained by its opposite sides.**

Let $ABCD$ be a quadrilateral inscribed in a circle;
join AC, BD;
then the rectangle $AC.BD$ shall be equal to the sum of the rectangles $AB.CD$ and $BC.AD$.



Construction.—At the point A in the straight line AB make the angle BAE equal to the angle DAC, and let AE meet BD in E.

Proof.—Because $\angle BAE = \angle DAC$, *Constr.*
add to each $\angle EAC$;
$\therefore$ $\angle BAC = \angle EAD$.
Then, in the two $\triangle$'s BAC and EAD,
because $\angle BAC = \angle EAD$, *Proved*
and $\angle ACB = \angle ADE$, in the same segment, *III. 21*
$\therefore$ $\triangle BAC$ is equiangular and therefore similar to $\triangle EAD$;
$\therefore$ $BC : AC :: ED : AD$; *VI. 4*
$\therefore$ the rectangle $BC.AD$ = the rectangle $AC.ED$. *VI. 16*
Again, in the two $\triangle$'s BAE and DAC,
because $\angle BAE = \angle DAC$,
and $\angle ABE = \angle ACD$, in the same segment, *III. 21*

$\therefore$ $\triangle BAE$ is equiangular and therefore similar to $\triangle DAC$;

$\therefore$ $AB : BE :: AC : CD$; *VI. 4*

$\therefore$ the rectangle $AB.CD$ = the rectangle $AC.BE$. *VI. 16*

But the rectangle $BC.AD$ = the rectangle $AC.ED$; *Proved*

$\therefore$ the sum of the rectangles $AB.CD$ and $BC.AD$ = the sum of the rectangles $AC.ED$ and $AC.BE$;

but the sum of the rectangles $AC.ED$ and $AC.BE$ = the rectangle $AC.BD$;

$\therefore$ the rectangle $AC.BD$ = the sum of the rectangles $AB.CD$ and $BC.AD$. Q. E. D.

EXERCISES.

1. The diagonals of a cyclic quadrilateral intersect at right angles. Show that the area of the sum of the rectangles of the opposite sides is double the area of the quadrilateral.

2. $ABCD$ is a cyclic quadrilateral, and the diagonal BD bisects AC. Prove that the rectangle $AB.AD$ is equal to the rectangle $CB.CD$.

3. ABC is an isosceles triangle, and from B and C, the ends of the base BC, perpendiculars to AB and AC are drawn, intersecting in O. Prove that the rectangle $BC.AO$ is double of the rectangle $AB.OB$.

4. State and prove the converse of Proposition D.

5. ABC is an equilateral triangle inscribed in a circle, and P is any point in the arc AB. Join PA, PB, and PC; and prove that PB is equal to the sum of PA and PC.

6. $ABCD$ is any quadrilateral. Make the angle CBE equal to the angle ABD, and the angle BCE equal to the angle ABD; and join AE. Hence prove that the rectangle $AC.BD$ is less than the sum of the rectangles $AB.CD$ and $AD.BC$, unless $ABCD$ is a cyclic quadrilateral.

Miscellaneous Exercises.

1. Divide a given finite straight line into two parts, such that their mean proportional may be equal to a given straight line.

2. Use Prop. 37 in Book III. to find a mean proportional between two given straight lines.

3. On a given finite straight line construct a rectangle equal to a given rectangle.

4. If a straight line AB be divided harmonically at P and Q, show that

(i.) the rectangle $AP.BQ$ = the rectangle $AQ.PB$;

(ii.) the rectangle $AB.PQ$ = twice the rectangle $AP.BQ$.

5. Deduce Prop. VI. 19 from Prop. VI. 23.

6. Parallelograms about the diagonal of any parallelogram are similar to each other.

7. In the figure of Prop. 24 prove that EG and BD are parallel.

8. Any two diagonals of a regular pentagon which intersect divide one another in extreme and mean ratio.

9. AB is divided at C so that the rectangle $AC.BC$ is equal to the rectangle contained by the sum and difference of AC and BC. Show that AB is divided in extreme and mean ratio at C.

10. If the radius of a circle be divided in extreme and mean ratio, the greater segment will be equal to a side of a regular decagon inscribed in the circle.

11. On AB as hypotenuse describe a right-angled triangle such that

$$AB : AC :: AC : BC.$$

12. Describe a regular pentagon equal in area to a regular hexagon.

13. Use the method of VI. 10 to divide a given straight line *externally* in extreme and mean ratio.

14. Divide a given straight line externally at a point C such that AB is a mean proportional between BC and AC.

15. Prove, by the method of superposition, that sectors in the same circle that have equal angles are equal in area.

16. Show that Euclid assumes the truth of Ex. 15 in Proposition 33.

17. In equal circles equal sectors stand on equal arcs.

18. If the exterior angle at A be bisected by a line AF, which meets the base BC produced in F, prove that the rectangle $AB.AC$ together with the square on AF is equal to the rectangle $BF.CF$.

19. Describe a triangle, having given the base, the rectangle contained by the sides, and the diameter of the circumscribed circle.

Notes on Propositions 21 to D.

Proposition 21.—An example of this theorem is found in VI. 8, where the triangles *ABD* and *ADC* are similar to the whole triangle *ABC* and to each other. Another example is found in Prop. 24, where parallelograms about the diagonal of any parallelogram are similar to the whole parallelogram and to each other.

Proposition 22.—Euclid proves this theorem by taking a third proportional to *AB* and *CD*, and another third proportional to *EF* and *GH*; and then using VI. 20, Cor. 1. This is a tedious method, and in the converse it is necessary to prove that "Equal rectilineal figures that are similar and similarly described have their homologous sides equal." The proof given in the text depends on V., Additional Prop. i.: that, if *AB* : *CD* :: *EF* : *GH*, then the duplicate ratio of *AB* to *CD* is the same as the duplicate ratio of *EF* to *GH*. This is evident in Algebra: for, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a^2}{b^2} = \frac{c^2}{d^2}$.

Proposition 23.—This proposition is difficult to a beginner, probably because he has not grasped the meaning of "compounding ratios" (see V., Def. 11 and Notes). Euclid proves the theorem for parallelograms; but it is better to prove it for triangles, and then its truth for parallelograms is an easy corollary.

If the triangles *ABC* and *BED* are similar, then *AB* : *BE* :: *CB* : *BD*, and the ratio compounded of *AB* to *BE* and *CB* to *BD* is the same as the duplicate ratio of *CB* : *CD*. This shows that VI. 19 is a special case of VI. 23. The proof that *CB* and *BD* are in the same straight line is omitted, because it is given in full in VI. 15. The proof of VI. 23 depends on Prop. 1.

Proposition 24.—Euclid's enunciation of this is: "Parallelograms about the diagonal of any parallelogram are similar to the whole parallelogram and to one another." The proof falls into two parts. In (i.) we prove that the parallelograms are equiangular; and in (ii.) we prove that their sides about the equal angles are proportional, by VI. 4.

Proposition 25.—*AB* and *BF* are in the same straight line because the angles at *B* are right angles. Prop. 25 is an easy problem, and might be stated familiarly thus: "To make a figure which shall have the form of one figure and the size of another."

Proposition 26.—This is the converse of Prop. 24, and might well have followed it immediately. Euclid's proof is by *reductio ad absurdum*; but the direct proof is easier and better.

Proposition 30.—A straight line divided in extreme and mean ratio is also said to be divided in *medial* section (see Book II., Prop. 11).

Proposition 31.—This is an extension of I. 47. It is an easy rider on VI. 8 and VI. 20, Cor. 1.

Proposition 33.—This proposition makes use of Euclid's definition of Proportion given in Book V. (see V., Def. 4), and its proof therefore resembles the proof of VI. 1. It is noticeable that VI. 33 makes no use of any proposition in Book VI., and it might therefore advantageously follow VI. 1, because the same method of proof is used in both propositions. In VI. 33, Euclid appears to disregard any restriction as to the size of an angle; for the angle *BGL* may be any multiple of the angle *BGC*, and may therefore be greater than two right angles.

Propositions B, C, and D.—These propositions were added by Simson, and they are all important theorems. The proofs of all of them depend on proving certain triangles equiangular and therefore similar. Prop. D is most important. It is found in the works of Ptolemy, who lived at Alexandria about 70 A.D. Ptolemy's Theorem is really a special case of the following theorem:—The rectangle contained by the diagonals of any quadrilateral is less than the sum of the rectangles contained by its opposite sides, unless the quadrilateral can be inscribed in a circle, in which case it is equal to their sum (see Ex. 6, page 69).

Summary of the Results arrived at in Euclid, Book VI.

Book VI. deals with Proportion applied to Geometrical Magnitudes.

Problems.

The Problems in Book VI. deal with either Lines or Rectilineal Figures.

I. *Problems dealing with Lines.*

(1) From a given finite straight line to cut off any required submultiple. (Prop. 9.)

(2) To divide a given finite straight line similarly to a given divided straight line. (Prop. 10.)

(3) To find a third proportional to two given finite straight lines. (Prop. 11.)

(4) To find a fourth proportional to three given finite straight lines. (Prop. 12.)

(5) To find a mean proportional between two given finite straight lines. (Prop. 13.)

(6) To divide a given straight line in extreme and mean ratio. (Prop. 30.)

II. *Problems dealing with Rectilineal Figures.*

(7) On a given straight line to describe a rectilineal figure similar and similarly described to a given rectilineal figure. (Prop. 18.)

(8) To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another given rectilineal figure. (Prop. 25.)

THEOREMS.

The Theorems in Book VI. may be arranged in three sets, treating of—

I. Segments of Lines;
II. Similar Figures;
III. Areas.

I. *The ratios of Segments of Lines.*

(1) A straight line parallel to one side of a triangle cuts the other sides proportionally; and conversely. (Prop. 2.)

(2) The bisector of the vertical angle of a triangle, either externally or internally, divides the base into segments which have the same ratio as the sides of the triangle; and conversely. (Props. 3 and A.)

II. *Similar Figures.*

(3) Two triangles—

(*a*) Are similar, if they are equiangular. (Prop. 4.)

(*b*) Are equiangular, if their sides taken in order are proportional. (Prop. 5.)

(*c*) Are similar, if angles, one in each, are equal and the sides about these equal angles are proportional. (Prop. 7.)

(*d*) Are equiangular or supplementary, if angles, one in each, are equal, and the sides about two other angles are proportional. (Prop. 7.)

(4) Rectilineal figures similar to the same rectilineal figure are similar to one another. (Prop. 21.)

(5) In a right-angled triangle, the triangles on each side of the perpendicular are similar to the whole triangle and to each other. (Prop. 8.)

(6) Any parallelogram about the diagonal of a parallelogram is similar to it; and conversely. (Props. 24 and 26.)

(7) Similar polygons may be divided into the same number of similar triangles. (Prop. 20, i.)

III. *The ratios of Areas.*

(8) Triangles and parallelograms of the same altitude are proportional to their bases. (Prop. 1.)

(9) In equal circles, angles, whether at the centres or circumferences, have the same ratio as the arcs on which they stand; so have their sectors. (Prop. 33.)

(10) Triangles and parallelograms equal in area, and which have a pair of equal angles, have their sides about the equal angles reciprocally proportional; and conversely. (Props. 15 and 14.)

(11) If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely. (Prop. 16.)

(12) If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and conversely. (Prop. 17.)

(13) Similar triangles and polygons are to each other in the duplicate ratio of corresponding sides. (Props. 19 and 20, iii.)

(14) If pairs of similar figures described on four straight lines be proportional, the four lines are proportional; and conversely. (Prop. 22.)

(15) Triangles and parallelograms having a pair of equal angles have to one another the ratio which is compounded of the ratios of the sides about the equal angles. (Prop. 23.)

(16) In a right-angled triangle any rectilineal figure on the hypotenuse is equal to the sum of similar and similarly described rectilineal figures on the other two sides. (Prop. 31.)

(17) The rectangle contained by the sides of a triangle is equal to—

(a) The rectangle contained by the segments of the base and the square on the bisector of the vertical angle. (Prop. B.)

(b) The rectangle contained by the perpendicular on the base and the diameter of the circum-circle. (Prop. C.)

(18) In a cyclic quadrilateral, the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides. (Prop. D.)

In any other quadrilateral the rectangle is less.

On Centres of Similitude.

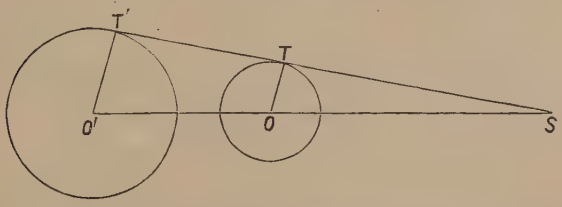
Definition.—If the line joining the centres of two circles be divided internally and externally in the ratio of the radii of the circles, the points of division are called, respectively, the internal and the external centres of similitude of the two circles.

1. *The direct common tangent of two circles passes through their external centre of similitude.*

Let $T'TS$ be a direct common tangent of the two circles whose centres are T' and T; join $O'O$, and produce $O'O$ to meet $T'T$ in S;

then S shall be the external centre of similitude of the two circles.

Join $O'T'$ and OT.



Because $\triangle$'s $SO'T'$ and SOT are equiangular and therefore similar,

$\therefore$ $O'T' : OT :: O'S : OS$;

$\therefore$ $O'O$ is divided externally in the ratio of the radii of the circles;

$\therefore$ S is the external centre of similitude of the two circles. Q. E. D.

Exercises on Centres of Similitude.

1. The transverse common tangent of two circles passes through the internal centre of similitude.

2. Where are the two centres of similitude of two circles—

(*a*) Of equal radii;

(*b*) If one circle cuts the other;

(*c*) If one circle is entirely inside the other circle?

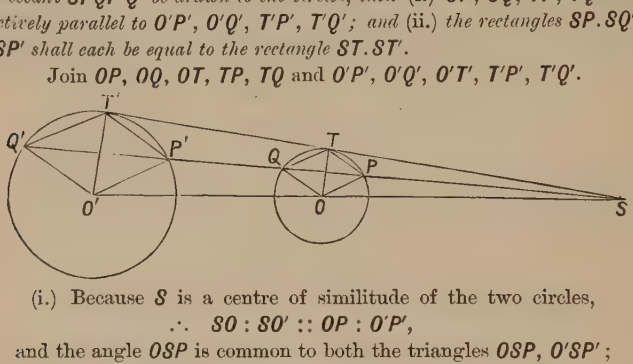
3. In the figure on page 76, prove that the rectangle $SP.SQ'$ is a fourth proportional to the radii of the two circles and the rectangle $SP'.SQ'$.

4. Every straight line which passes through the extremities of two parallel radii of two given circles also passes through a centre of similitude of the circles.

5. A circle touches two given circles. Show that the straight line which passes through the points of contact also passes through a centre of similitude of the two circles.

2. *If through* S, *a centre of similitude of two circles, a common tangent* STT' *and a secant* $SPQP'Q'$ *be drawn to the circles, then* (i.) OP, OQ, TP, TQ *shall be respectively parallel to* $O'P'$, $O'Q'$, $T'P'$, $T'Q'$; *and* (ii.) *the rectangles* $SP.SQ'$ *and* $SQ.SP'$ *shall each be equal to the rectangle* $ST.ST'$.

Join OP, OQ, OT, TP, TQ and $O'P'$, $O'Q'$, $O'T'$, $T'P'$, $T'Q'$.



(i.) Because S is a centre of similitude of the two circles,

$\therefore$ $SO : SO' :: OP : O'P'$,

and the angle OSP is common to both the triangles OSP, $O'SP'$;

$\therefore$ $\triangle$'s OSP, $O'SP'$ are similar and equiangular; *VI. 7*

$\therefore$ OP is parallel to $O'P'$.

Similarly it can be proved that OQ is parallel to $O'Q'$.

Again, because $SP : SP' :: SO : SO'$

$:: ST : ST'$, by former theorem (page 75),

$\therefore$ $\triangle$'s STP, $ST'P'$ are similar and equiangular; *VI. 7*

$\therefore$ TP is parallel to $T'P'$.

Similarly it can be proved that TQ is parallel to $T'Q'$.

(ii.) Because $SP.SQ = ST^2$, *III. 36*

$\therefore$ $SP : ST :: ST : SQ$

$:: ST' : SQ'$, by (i.);

$\therefore$ $SP.SQ' = ST.ST'$.

Similarly it can be proved that $SQ.SP' = ST.ST'$. Q.E.D.

EXERCISES (*continued*).

6. If two unequal similar triangles OPT and $O'P'T'$ be placed with corresponding sides parallel, prove that the three straight lines joining corresponding vertices are concurrent.

7. In the last example, show that there are two possible points of concurrence, and that these two points divide each joining line internally and externally in the ratio of any pairs of homologous sides.

8. Extend Exx. 7 and 8 to unequal similar polygons of any number of sides.

9. The orthocentre and the centroid of a triangle are the centres of direct and inverse similitude of the circum-circle and nine-point-circle of the triangle.

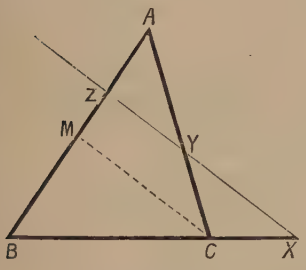
10. The six centres of similitude of three circles lie three by three on four straight lines, called the "axes of similitude" of the circles.

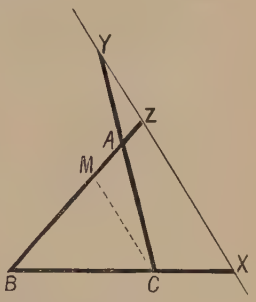
On Transversals.

Definition.—A *Transversal* is a straight line drawn to cut a series of lines.

1. *If a transversal be drawn to cut the three sides of a triangle, produced if necessary, then the product of three alternate segments taken in order is equal to the product of the other three segments.* (Theorem of Menelaus.)

Let the transversal XY be drawn to cut the three sides of the triangle ABC, viz., BC at X, CA at Y, and AB at Z, the sides being produced, if necessary; then shall $BX.CY.AZ = XC.YA.ZB$.





Construction.—Draw CM parallel to XY, meeting AB in M.

Proof.—Because XZ is parallel to CM, a side of $\triangle BCM$,

$$\therefore \quad \frac{BX}{XC} = \frac{BZ}{ZM}; \qquad \textit{VI. 2}$$

and, because XY is parallel to CM, a side of $\triangle ACM$,

$$\therefore \quad \frac{CY}{YA} = \frac{MZ}{ZA}, \qquad \textit{VI. 2}$$

and $$\frac{AZ}{ZB} = \frac{AZ}{ZB};$$

$\therefore$ by multiplication, $$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{BZ}{ZM}\cdot\frac{MZ}{ZA}\cdot\frac{AZ}{ZB} = 1;$$

$$\therefore \quad BX.CY.AZ = XC.YA.ZB. \qquad \text{Q. E. D.}$$

This theorem is said to be due to Menelaus, a Greek, who lived towards the end of the first century. Its converse is also true, and may easily be proved indirectly.

It is evident that any transversal XY will cut three sides of a triangle ABC when one side is produced as in Fig. 1, or when all three sides are produced as in Fig. 2.

Menelaus' Theorem may also be enunciated as follows:—

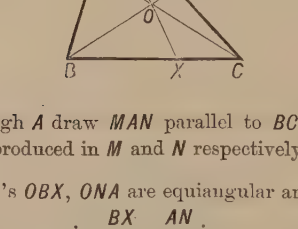
If a transversal cut the sides of a triangle ABC *in the points* X, Y, Z, *then the product of the ratios* $\frac{BX}{XC}, \frac{CY}{YA}, \frac{AZ}{ZB} = -1$.

For, since the transversal cuts either one or three of the sides produced, either one or three of these ratios will be negative, if we use the terms *plus* and *minus* as explained on page 114, Book II.; and therefore the product of the ratios will be negative, and therefore equal to -1.

2. *If three concurrent straight lines be drawn from the angular points of a triangle to meet the opposite sides, produced if necessary, then the product of three alternate segments taken in order is equal to the product of the other three segments.* (Ceva's Theorem.)

Let the three straight lines AX, BY, CZ be drawn from the angular points of the triangle ABC, so as to intersect in O and to meet the opposite sides, produced if necessary, in X, Y, Z;

then shall $BX.CY.AZ = XC.YA.ZB$.



Construction.—Through A draw MAN parallel to BC, to meet CZ and BY produced in M and N respectively.

Proof.—Because $\triangle$'s OBX, ONA are equiangular and therefore similar,

$$\therefore \frac{BX}{OX} = \frac{AN}{AO}; \qquad VI.\ 4$$

and because $\triangle$'s OCX, OMA are equiangular and therefore similar,

$$\therefore \frac{XC}{OX} = \frac{MA}{AO}; \qquad VI.\ 4$$

$$\therefore \frac{BX}{XC} = \frac{AN}{MA} \ldots\ldots (1).$$

Again, because $\triangle$'s YBC, YNA are equiangular and therefore similar

$$\therefore \frac{CY}{YA} = \frac{BC}{AN} \ldots\ldots (2). \qquad VI.\ 4$$

And, because $\triangle$'s ZAM, ZBC are equiangular and therefore similar,

$$\therefore \frac{AZ}{ZB} = \frac{MA}{BC} \quad \ldots\ldots (3). \qquad VI.\ 4$$

$\therefore$ multiplying together (1), (2), (3),

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AN}{MA} \cdot \frac{BC}{AN} \cdot \frac{MA}{BC} = 1.$$

$$\therefore BX.CY.AZ = XC.YA.ZB. \qquad \text{Q. E. D.}$$

This theorem was first published by Ceva, an Italian, in 1678. Its converse is also true, and may easily be proved indirectly.

The point O may lie either within or without the triangle ABC, and the lines AO, BO, CO will cut the sides of the triangle ABC either without producing any of the sides, as in the figure on page 78, or when two sides of the triangle ABC are produced.

Ceva's Theorem may also be enunciated thus:—

If three concurrent straight lines AO, BO, CO *be drawn from the angular points of the triangle* ABC *to meet the opposite sides, produced if necessary, then the product of the ratios* $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

For, since the lines AO, BO, CO cut the sides when either none or two of the sides are produced, either none or two of these ratios will be negative; and therefore the product of the ratios will be positive, and therefore equal to 1.

Exercises on Transversals.

1. Prove Menelaus' Theorem by similar triangles.

2. Prove Menelaus' Theorem by drawing perpendiculars from A, B, C on the transversal XYZ, taking similar triangles.

3. Prove Props. 5, 7, and 8 on pages 101, 102, Book I., by using the converse of Ceva's Theorem.

4. Prove Ceva's Theorem by using Ex. 2, page 15 (VI. 1).

5. The straight lines which join the vertices of a triangle to the points of contact of the inscribed circle are concurrent. Is this true for an escribed circle?

6. The middle points of the diagonals of a complete quadrilateral are collinear.

7. State and prove the converse of Menelaus' Theorem.

8. State and prove the converse of Ceva's Theorem.

9. Prove that the external bisectors of the angles of a triangle meet the opposite sides in collinear points.

10. Tangents to the circum-circle at the vertices of the inscribed triangle meet the opposite sides in collinear points.

11. In Ceva's Theorem, if X bisects BC, show that YZ is parallel to BC.

On Harmonic Section.

Before reading this section the student should make himself quite familiar with VI., Defs. 7 and 9, and Props. 3 and A and the Exercises on those Propositions.

1. *If P, Q be harmonic conjugates with respect to A, B, and AB be bisected in O, then the rectangle $OP.OQ$ shall be equal to the square on OB.*

Because $AP : PB :: AQ : BQ$,

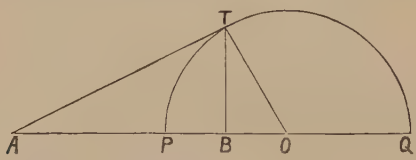
$\therefore\ AP - PB : AP + PB :: AQ - BQ : AQ + BQ$;

$\therefore\ 2OP : 2OB :: 2OB : 2OQ$;

$\therefore\ OP.OQ = OB^2$. Q. E. D.

The converse of this theorem is true.

2. *The geometric mean between two straight lines is the mean proportional between their arithmetic and harmonic means.*



Let AP and AQ represent the two straight lines.

Bisect PQ at O, and on PQ describe the semi-circle PTQ;

draw the tangent AT, and TB perpendicular to PQ.

Then (a) $\dfrac{AP + AQ}{2} = AO$ = arithmetic mean of AP and AQ;

(b) $AT^2 = AP.AQ$;

$\therefore\ AT$ is the geometric mean of AP and AQ;

(c) $OB.OA = OT^2 = OP^2$;

$\therefore$ by converse of Theorem 1 (above), A, B, P, Q is a harmonic range;

$\therefore\ AB$ is the harmonic mean between AP and AQ.

Because ATO is a right-angled triangle and TB perpendicular to AO,

$\therefore\ AT^2 = AO.AB$. Q. E. D.

Definitions.

1. A *range* is a series of points in a straight line.

2. A *pencil* is a series of straight lines passing through one point, called the *vertex*.

3. A *ray* is one of the straight lines in a pencil.

4. A *harmonic pencil* is a pencil of rays which pass through the four points of a harmonic range.

$(O.ABCD)$ or $O\,(ABCD)$ denotes a harmonic pencil whose vertex is O.

Exercises on Harmonic Section.

1. If A, P, B, Q be a harmonic range, the circles described on AB and PQ as diameters will cut each other orthogonally, that is, at right angles.

2. Show that the geometric definition of harmonic mean coincides with the algebraic definition.

3. Given the base of a triangle and the ratio of the other sides, show that the locus of the vertex is a circle. (Theorem of Apollonius.)

4. Any straight line passing through a fixed point and cutting a circle is divided harmonically by the circle, the point, and the polar of the point.

5. If a transversal be drawn parallel to one ray of a harmonic pencil, the other three rays will intercept equal parts on the transversal; and conversely.

6. By means of Ex. 5 show that every transversal is cut harmonically by the rays of a harmonic pencil.

7. If one pair of rays of a harmonic pencil are at right angles to each other, then these rays are the internal and external bisectors of the angle between the other pair of rays.

8. If one ray of a harmonic pencil bisect the angle between two other rays, then this ray will be perpendicular to its conjugate ray.

9. Any transversal through the middle point of the base of a triangle is divided harmonically by the sides of the triangle and the straight line through the vertex parallel to the base.

10. If $ABCD$ and $Abcd$ be two harmonic ranges having the common point A, then Bb, Cc, and Dd will be concurrent.

11. $(O.APBQ)$ is harmonic if a transversal parallel to any one ray is divided into equal segments by the remaining three rays.

12. Find a fourth harmonic point to three given points.

13. Draw a fourth harmonic ray to three given rays.

14. $ABCD$ is a complete quadrilateral, and AB and BC intersect in X, AB and DC in Y, and AC and BD in Z. Show that XZ and XY are harmonic conjugates with respect to XA and XB.

15. In Ex. 14, if XZ, YZ, and AZ be produced to meet opposite lines in the figure, write down all the harmonic ranges which the complete figure contains.

Miscellaneous Exercises and Riders on Book VI.

I.—*On the Ratios of Segments of Lines.*

(Props. 2, 3, A, 9, 10, 11, 12, 13, and 30.)

1. $ABCD$ is a quadrilateral, and P, Q, R, S are points on AB, BC, CD, DA such that $AP : AB :: CQ : CB :: CR : CD :: AS : AD$.
Prove that $PQRS$ is a parallelogram.

2. ABC is an isosceles triangle having the side AB equal to the side AC, and the external angle at A is bisected by AD. Where will AD meet BC?

3. If any straight line DE be drawn parallel to BC, one of the sides of the triangle ABC, and meeting AB and AC in D and E, and if DE be divided at P in a given ratio, show that the locus of P is a straight line.

4. The line joining any point P on a given circle to a fixed point Q is divided in a given ratio at R. Prove that the locus of R is a circle.

5. Show that the mean proportional between two given straight lines is less than half their sum.

6. Divide a given arc of a circle into two parts so that the chords of these parts may be to each other in a given ratio.

7. If the sides of a right-angled triangle are in continued proportion, then the straight line from the right angle perpendicular to the hypothenuse will divide the hypothenuse in extreme and mean ratio.

8. ABC is a right-angled triangle, having ACB the right angle, and AD bisecting the angle BAC, cutting BC in D. Show that

$$AB + AC : BC :: AC : CD.$$

9. In Ex. 8, if BE be drawn perpendicular to AD, show that

$$AB + AC : BC :: AE : EB.$$

10. If PQR be a triangle inscribed in the triangle ABC, and having its sides parallel to the sides of ABC, then P, Q, R must be the middle points of the sides of ABC.

11. If the bisectors of one pair of opposite angles of a quadrilateral meet on one diagonal, then the bisectors of the other pair of angles will meet on the other diagonal.

12. If the hypothenuse of a right-angled triangle is divided in extreme and mean ratio by the perpendicular drawn to the hypothenuse, prove that (1) the lesser side of the triangle is equal to the larger segment of the hypothenuse; (2) the sides of the triangle are in continued proportion.

13. From a fixed point O straight lines are drawn, and in them points X and Y are taken, such that the ratio of OX to OY is constant. Prove that (1), if the locus of either X or Y is a straight line, the locus of the other point is also a straight line; and (2), if the locus of either X or Y is a circle, the locus of the other point is also a circle.

14. A, B, C, D are collinear points, and on AC and BD two similar triangles APB and CQD are described, so that the corresponding sides AP and BQ, and again CP and DQ, are parallel. Then, if QP and DA intersect in O, prove that

$$OA : OB :: OC : OD.$$

II.—*On Similar Figures.*

(Props. 4, 5, 6, 7, 8, 18, 20, 21, 24, and 26.)

15. If the sides of a triangle be cut by a straight line parallel to the base, the triangular portion cut off is similar to the whole triangle.

16. From a given triangle cut off a similar triangle by a straight line which is *not* parallel to any side.

When is this impossible?

17. The ratios of the intercepts made on two parallel lines by any number of straight lines meeting in a point is constant.

18. If two straight lines AB, CD intercept in O, so that $AO : OC :: DO : OB$, prove that A, B, C, D are concyclic.

19. The straight lines which join either directly or transversely the extremities of two parallel diameters of any two circles intersect on the line which joins the centres of the circles.

20. OP and OQ are two radii of a circle at right angles. Draw a chord of the circle which is trisected by OP and OQ.

21. On AC, the side of a right-angled triangle ABC, a semi-circle is described cutting the hypothenuse BC in D. Prove that the tangent at D bisects AB.

22. The tangent at A to a circle whose centre is O meets two parallel tangents at B and C. Show that OA is a mean proportional between OB and OC.

23. Two circles intersect in P and Q, and through P a straight line is drawn cutting the circles in A and B. Show that the locus of the middle point of AB is a circle.

24. The perimeters of similar rectilineal figures are to one another in the ratio of corresponding sides.

25. From the point of intersection of the diagonals of a cyclic quadrilatera perpendiculars are dropped on a pair of opposite sides. Prove that these perpendiculars are in the same ratio as the sides to which they are drawn.

26. Given two finite straight lines AB and CD, find a point P which shall be the common vertex of two similar triangles PAB and PCD.

III.—*On the Ratios of Areas.*

(Props. 1, 14, 15, 16, 17, 19, 20, 23, 25, 31, 33, B, C, and D.)

27. $ABCD$ is a quadrilateral whose diagonals meet in E. Show that, if the triangles AEB, CED are equal in area, then the triangles AED, CEB are similar.

28. If A, B, C, D be four points in order on a straight line, find a point O in the line such that the rectangle $OA.OD$ is equal to the rectangle $OB.OC$.

29. The ratio of the areas of two isosceles triangles is equal to the duplicate ratio of their bases. Show that the triangles are similar.

30. Construct an equilateral triangle which shall be equal in area to a given quadrilateral.

31. Bisect a triangle by a straight line at right angles to one of the sides.

32. ABC is a triangle having BAC a right angle, and AD is drawn perpendicular to BC. Prove that

$$\text{sq. on } BC : \text{sq. on } BA :: BC : BD.$$

33. Similar triangles are to one another in the duplicate ratio of (*a*) corresponding medians, (*b*) the radii of their inscribed circles, (*c*) the radii of their circumscribed circles.

34. If the vertex A of a triangle ABC be joined to any point D in the base BC, then the two triangles ADB and ADC will be such that the radii of their circumscribed circles are in the ratio of AB to AC.

35. The ratio of the areas of two triangles is the ratio compounded of the ratio of their bases and the ratio of their altitudes.

36. ABC is any triangle, and BE, CF are drawn perpendicular to AC, AB, respectively. Prove that

$$\triangle ABC : \triangle ACF :: \text{sq. on } AB : \text{sq. on } AC.$$

37. Triangles which have one angle of the one supplementary to one angle of the other and their sides about the supplementary angles reciprocally proportional are equal in area.

38. If two triangles are on opposite sides of the same base, their areas are proportional to the segments of the straight line which joins their vertices.

39. $ABCD$ is a parallelogram, and a line is drawn from A cutting BD in P, CD in Q, and BC produced in R. Prove that $PQ : PR$ in the duplicate ratio of $PD : PB$.

40. $ABCDE$ is a regular pentagon inscribed in a circle, and P is any point on the arc AE. Prove that $PA + PC + PE = PB + PD$.

41. $ABCD$ is a parallelogram, and a circle cuts AB, AC, AD in P, Q, R respectively. Prove that $AB.AP + AD.AR = AC.AQ$.

Harder Theorems.

42. Find a point the perpendiculars from which, on the sides of a given triangle, shall be in a given ratio.

43. Given three points A, B, C on a straight line, find the locus of points at which AB and BC subtend equal angles.

44. The bisector of the angle BAC of a triangle meets BC in D, and meets in E the straight line which bisects BC at right angles. Prove that

$$EA : EB :: EB : ED.$$

45. If X and Y be the points of intersection of a variable circle, drawn through two fixed points A and B with a given circle, then the ratio of the rectangle $AX.AY$ to the rectangle $BX.BY$ is constant.

46. Two circles intersect at P, and the chords PQ and PR are also tangents at P. Join QR and show that it is divided by the common chord of the two circles into segments which are to one another in the duplicate ratio of PQ to PR.

47. Given the side BC of a triangle, the radius of the circum-circle, and the radius of the circle escribed to BC, construct the triangle.

48. About a given quadrilateral describe a quadrilateral similar to another given quadrilateral.

49. A variable tangent to a circle meets two fixed tangents in P and Q. Find the position of the variable tangent when the intercept PQ is the least.

50. Given the base of a triangle, the ratio of the other sides, and that the vertex lies on a given straight line, construct the triangle.

When is this impossible?

51. The areas of similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.

52. The ratio of twice the area of a triangle to the rectangle contained by two of the sides is equal to the ratio of the third side to the diameter of the circumscribed circle of the triangle.

53. Divide a triangle into 3 equal parts by straight lines parallel to the base.

54. Bisect a triangle by a straight line drawn through a given point (*a*) within, (*b*) without it.

55. *ABCD* is a cyclic quadrilateral. Prove that

$$AB.AD + CB.CD : BA.BC + DA.DC :: AC : BD.$$

56. *ABC* is an isosceles triangle. Show that the locus of a point *P*, which moves so that the rectangle under its distances from the equal sides *AB* and *AC* is equal to the square on its distance from the base *BC*, is the circle which touches *AB* and *AC* at *B* and *C*.

57. The locus of a point which moves so that the rectangle under its distances from a pair of opposite sides of a cyclic quadrilateral is equal to the rectangle under its distances from the other pair of sides is the circle circumscribing the quadrilateral.

58. *O*, *A*, *B*, *C*, *D* are collinear points such that the rectangle *OA.OD* is equal to the rectangle *OB.OC*, and a circle is described with centre *O* and the square of whose radius is equal to *OA.OD* or *OB.OC*. Show that the lines *AB* and *CD* subtend equal angles at any point on the circumference of this circle.

59. *ABC* is any triangle, and equal straight lines *AX*, *BY*, *CZ* are drawn to meet the opposite sides of *ABC*. If, from any point *P* within *ABC*, *PL*, *PM*, *PN* are drawn to meet the sides of *ABC* parallel to *AX*, *BY*, *CZ*, then

$$AX = PL + PM + PN.$$

60. If the triangle *DEF* is inscribed in the triangle *ABC*, so that each pair of sides of *DEF* make equal angles with the conterminous side of *ABC*, then *DEF* shall be the pedal triangle of *ABC*.

61. Given the sides and angles of a triangle and that two of its sides pass through two fixed points, prove that the third side will touch a fixed circle.

62. The middle points of the three diagonals of a complete quadrilateral are collinear.

63. If a triangle of given species has one angle fixed, and another angle moves along a given straight line, then the third angle lies on a fixed line.

64. If a triangle of given species has one angle fixed, and another angle moves along the circumference of a given circle, then the third angle will always lie on the circumference of a fixed circle.

65. *ABCD* is a complete quadrilateral, and *AC*, *BD* intersect in *Z*; then the straight line *ZP* drawn through *Z* parallel to *AB* to meet the third diagonal in *P* will be bisected by *DC* produced.

ON REFORMS IN TEACHING GEOMETRY SUGGESTED BY A COMMITTEE OF THE MATHEMATICAL ASSOCIATION.

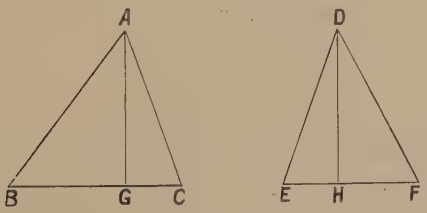
The following suggestions, dealing with Euclid, Book VI., are taken from the recommendations made by a Committee of the Mathematical Association in 1902 :—

1. The study of Book VI. should be preceded by—
   (i.) A theory of measurement of lengths of lines and areas of rectangles ;
   (ii.) An algebraical treatment of ratio and proportion for commensurables.

2. In an ordinary school course, all magnitudes of the same kind should be treated as commensurable, and then Prop. 1 may be proved as follows :—

Let ABC, DEF be two triangles whose bases BC and EF are commensurable, and whose heights AG and DH are equal ;

then $\frac{\triangle ABC}{\triangle DEF}$ shall be equal to $\frac{BC}{EF}$;



$\therefore$ the area of a triangle is measured by half the product of the measures of its base and height (see Euclid, Book I., Prop. 41, page 97),

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC.AG}{\frac{1}{2}EF.DH}$$

$$= \frac{BC.AG}{EF.DH}$$

$$= \frac{BC}{EF}, \text{ since } AG = DH. \qquad \text{Q. E. D.}$$

3. In stating the conditions for the similarity of two triangles ABC, DEF,

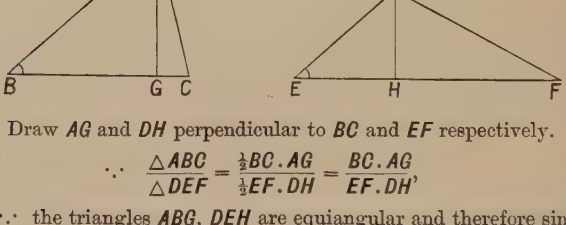
$$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE} \text{ or } BC : EF = CA : FD = AB : DE$$

is preferable to $\frac{AB}{BC} = \frac{DE}{EF}$, $\frac{BC}{CA} = \frac{EF}{FD}$, and $\frac{CA}{AB} = \frac{FD}{DE}$.

4. The expression *corresponding sides* is preferable to the expression *homologous sides* (see Book V., Def. 6, page 4).

5. Additional Proposition.—*If two triangles (or parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of the rectangles contained by the sides about the equal angles.*

Let ABC, DEF be two triangles, having $\angle B = \angle E$; then their areas are in the ratio $\frac{BC.AB}{EF.DE}$.



Draw AG and DH perpendicular to BC and EF respectively.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC.AG}{\frac{1}{2}EF.DH} = \frac{BC.AG}{EF.DH'}$$

and $\therefore$ the triangles ABG, DEH are equiangular and therefore similar;

$$\therefore \frac{AG}{DH} = \frac{AB}{DE};$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC.AB}{EF.DE}. \qquad \text{Q. E. D.}$$

6. From this Additional Proposition it follows that *the areas of similar triangles are proportional to the squares on corresponding sides.* (Euclid VI. 19 is thus proved.)

7. VI. 20 can be deduced from the Additional Proposition, and VI. 22 easily follows.

8. Props. 14, 15, 16, 17, 21, 23, 24, 26, 27, 28, 29, 32, Book VI., should be omitted.

9. All statements of ratio may be made in fractional form, and the sign = used instead of ::.

10. In the ordinary school course, reciprocal proportion should be dropped, and *compounding* replaced by *multiplying*.

EUCLID'S ELEMENTS.

BOOK XI.

BOOK XI.

ON SOLID GEOMETRY.

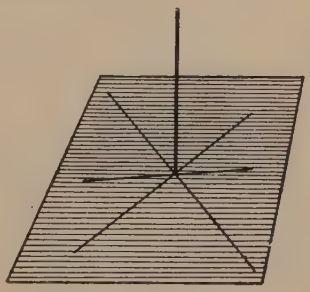
Definitions.

For definitions of a *line*, a *straight line*, a *surface*, a *plane surface* or *plane*, a *plane angle*, a *plane rectilineal angle*, a *perpendicular*, and *parallel straight lines*, see Definitions 2, 4, 5, 7, 8, 9, 10, and 30 in Book I.

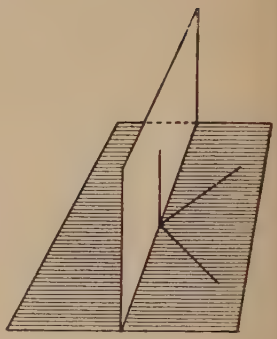
1. A *solid* has length, breadth, and thickness.

2. That which bounds a solid is a *surface*.

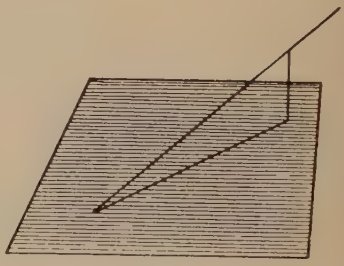
3. A *straight line is perpendicular to a plane* when it is perpendicular to every straight line which meets it in that plane.



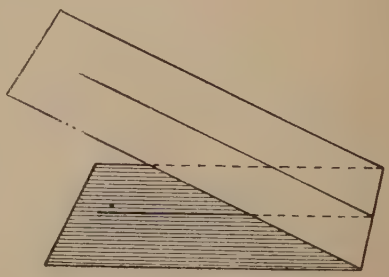
The point at which a perpendicular to a plane meets the plane is called the *foot of the perpendicular*.



4. One *plane is perpendicular to another plane* when every straight line drawn in the first plane perpendicular to the common section of the two planes is also perpendicular to the second plane.



5. *The inclination of a straight line to a plane* is the *acute* angle contained by that straight line and another drawn from the point at which the first line meets the plane to the foot of a perpendicular let fall from any point in the first line to the plane.



6. The *inclination of a plane to a plane* is the *acute* angle contained by two straight lines drawn from any point in the common section at right angles to it, one in one plane and the other in the other plane.

Notes on the Definitions.

1. This definition implies that space has three dimensions, viz.: length, breadth, and thickness.

3. A straight line perpendicular to a plane is often called a *normal* to the plane.

In XI. 4 it is proved that, if a straight line is perpendicular to two straight lines which meet it in a plane, it is perpendicular to every straight line which meets it in that plane. Euclid's definition, therefore, is redundant.

The foot of the perpendicular is often spoken of as the projection on the plane of the point from which the perpendicular is drawn. The *projection of a line on a plane* is the locus of the feet of the perpendiculars drawn from all points in the given line to the plane. (See Book I., page 100, 3, "Orthogonal Projection.")

4. Unless the contrary is stated, it is always assumed that straight lines and planes may be produced indefinitely. A line will intersect a plane in one point, unless it is parallel to the plane; and two planes will intersect in a line, unless they are parallel. The line in which two planes intersect is called their *common section*, and in XI. 3 it is shown that the common section of two planes is a straight line.

5. The projection of a straight line on a plane is a straight line (see Book I., page 100), and the *inclination of a line to a plane* is the angle between the line and its projection on the plane.

6. The angle formed by two planes is called a *dihedral angle*. If two adjacent dihedral angles made by two planes at a point are equal, the planes are at right angles to each other.

Euclid assumes in Def. 6 that the dihedral angle between the same two planes is of constant size. This is proved in XI. 10.

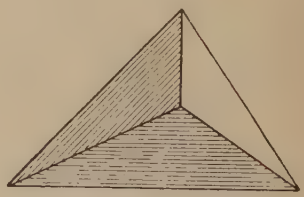
As in Trigonometry an angle is defined by the rotation of a line about a fixed point, so too a dihedral angle may be defined by the rotation of a plane about two fixed points.

The student should notice that the inclination of two planes is measured by a plane angle, and plane angles are measured by the arcs of circles (VI. 33)—that is, plane angles are measured by the *length* of an arc, but dihedral and solid angles (Def. 9) are measured by *areas*.

7. Two planes are said to have *the same inclination* to one another which two other planes have when their angles of inclination are equal.

8. *Parallel planes* are such as do not meet when produced.

9. A *solid angle* is that which is made by three or more plane angles which have a common vertex but are not in the same plane.



10. A *polyhedron* is a portion of space bounded on all sides by planes.

11. *Similar polyhedra* are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

9. A solid angle formed by three plane angles is called a *trihedral* angle; a solid angle formed by more than three plane angles is called a *polyhedral* angle.

A solid angle is also called a *corner*.

A dihedral angle is sometimes spoken of as a solid angle; and occasionally the words *tetrahedral* and *pentahedral* are used of solid angles formed by four or five plane angles respectively.

Re-entrant solid angles are such as are divided into two or more portions when one or other of their bounding planes is produced. *Convex* or *non-re-entrant* solid angles are such as are not divided into two or more portions when any of their bounding planes are produced.

A *right solid angle* is a trihedral angle formed by three planes which are at right angles to each other.

10. A *solid figure* is a portion of space bounded by one or more surfaces, either plane or curved.

The planes which bound a polyhedron are called the *faces* of the polyhedron; the line in which any two faces meet is called an *edge* of the polyhedron; and the point where three or more edges meet is called a *vertex*, or a *corner*, or an *angular point* of the polyhedron.

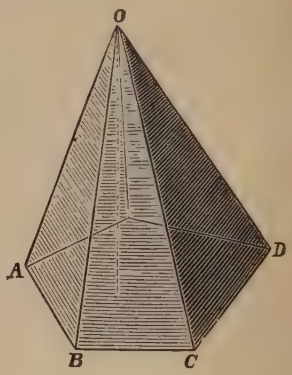
A polyhedron must have at least four faces.

A *tetrahedron* (τέτταρες, "four") is a polyhedron having 4 faces.
A *pentahedron* (πέντε, "five") ,, ,, ,, 5 ,,
A *hexahedron* (ἕξ, "six") ,, ,, ,, 6 ,,
A *octahedron* (ὀκτώ, "eight") ,, ,, ,, 8 ,,
A *dodecahedron* (δώδεκα, "twelve") ,, ,, ,, 12 ,,
An *icosahedron* (εἴκοσι, "twenty") ,, ,, ,, 20 ,,

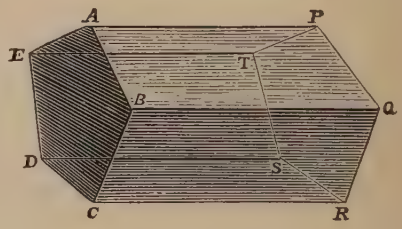
A *regular polyhedron* or *solid* is a polyhedron or solid of which all the faces are equal and similar regular polygons, and all the solid angles are equal.

Euclid's 10th definition is as follows:—"Equal and similar solid figures are such as are contained by similar planes equal in number and magnitude." This definition is not true. For two pyramids on the same base but of different altitudes will form two *unequal* solid figures if we place the bases exactly on each other, but place the vertices first on the same side and then on different sides of the base. Euclid's definition is true of *convex* solid figures. It seems better, however, to omit the definition, as Simson does, and to introduce the definition of a polyhedron. We are then able to use the term "solid figure" for a figure bounded by all kinds of surfaces, plane or curved, and to keep the word "polyhedron" for a solid figure bounded by planes only.

12. A *pyramid* is a solid figure contained by planes, of which all but one, the base, pass through the same point, the vertex.



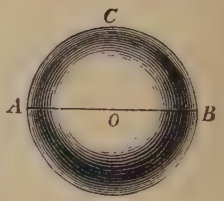
13. A *prism* is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel, and the others are parallelograms.



Solids of Revolution.

14. A *sphere* is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

15. The *axis* of a sphere is the fixed straight line about which the semicircle revolves.



16. The *centre* of a sphere is the same as that of the semicircle.

17. A *diameter* of a sphere is any straight line which passes through the centre, and is terminated both ways by the surface of the sphere.

12. A *pyramid* is a polyhedron whose faces, all but one, meet in a point.

A pyramid on a triangular base is a tetrahedron.

The altitude of a pyramid is the perpendicular distance of the vertex from the base of the pyramid.

13. A *prism* is a polyhedron whose faces, all but two, are parallel to one straight line.

The *ends* or *bases* of a prism are the two equal, similar, and parallel faces that are opposite each other; the other faces of the prism are called *sides*.

A *right prism* is a prism in which the two ends are at right angles to all the other faces; otherwise the prism is *oblique*. In a right prism the sides are all rectangles.

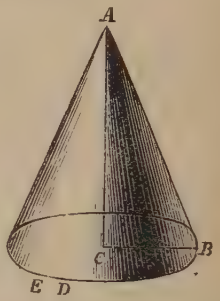
The *altitude of a prism* is the perpendicular distance between the ends or bases of the prism.

14. In defining a sphere, we might imitate the definition of a circle (see Book I., Def. 15), and say that a sphere is a solid contained by a surface, and is such that all straight lines drawn from a fixed point within it to the surface are equal.

A *radius of a sphere* is any straight line drawn from the centre of the sphere to the surface of the sphere.

A *frustrum*, or *frustum*, *of a sphere* is the part of a sphere intercepted between two parallel planes.

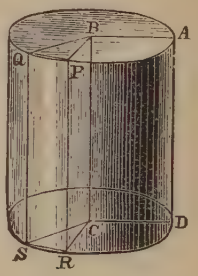
18. A *right circular cone* is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.



19. The *axis* of a cone is the fixed straight line about which the triangle revolves.

20. The *base* of a cone is the circle described by that side containing the right angle which revolves.

21. A *right circular cylinder* is the solid figure described by the revolution of a rectangle about one of its sides which remains fixed.



22. The *axis* of a cylinder is the fixed straight line about which the rectangle revolves.

23. The *bases* or *ends* of a cylinder are the circles described by the opposite sides of the rectangle which revolve.

24. *Similar cones and cylinders* are those which have their axes and the diameters of their bases proportionals.

18. Euclid omits the words "right circular" in Def. 18.

Geometricians also use a definition of a cone such as the following:—"A *conical surface in general* is a surface generated by a straight line which revolves so as always to pass through one fixed point and through some movable point on a curve in space."

The fixed point is called the *vertex*, and the vertex is not situated in the same plane as the guiding curve. This curve may even be rectilinear, or partly rectilinear and partly curvilinear. The revolving line is called the *generating line* of the cone. In Euclid's definition the hypothenuse is the generating line. The hypothenuse is also spoken of as the *slant side* of the cone.

The *altitude of a cone* is the perpendicular distance of the vertex from the base of the cone.

A cone is called a right-angled, an obtuse-angled, or an acute-angled cone according as the fixed side of the revolving triangle is equal to, less than, or greater than the other side containing the right angle. In these three cases the angle at the vertex is right, obtuse, or acute, respectively.

A *right cone* is a cone in which the straight line joining the vertex to the centre of the base is perpendicular to the base.

An *oblique cone* is a cone in which the straight line joining the vertex to the centre of the base is not perpendicular to the base. An oblique cone may be considered as a part of a right cone which is cut off by a plane not parallel to the base.

A *frustrum*, or *frustum, of a cone* is a truncated cone—that is, it is the part of a cone intercepted between two planes parallel to the base.

A *circular cone* is a cone in which the base is a circle.

21. Euclid omits the words "right circular" in Def. 21.

Geometricians also use a definition of a cylinder such as the following:—"A *cylindrical surface in general* is a surface generated by a straight line which moves parallel to a fixed direction and through some movable point on a curve in space."

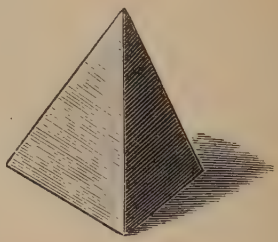
The line which moves is called the *generating line* of the cylinder. In Euclid's definition the generating line is the side of the rectangle parallel to the fixed side.

The *altitude of a cylinder* is the perpendicular distance between its bases.

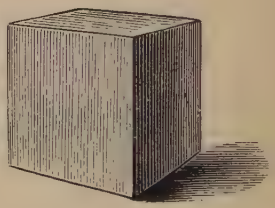
The words "right," "oblique," and "circular" are applied to cylinders with the same meanings as when applied to cones.

Regular Polyhedra.

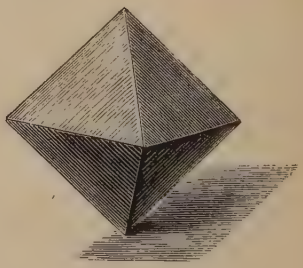
25. A *regular tetrahedron* is a solid figure contained by four equal equilateral triangles.



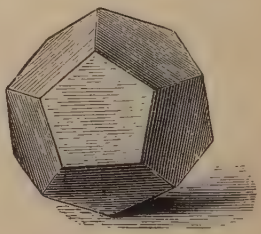
26. A *cube* is a solid figure contained by six equal squares.



27. A *regular octahedron* is a solid figure contained by eight equal equilateral triangles.



28. A *regular dodecahedron* is a solid figure contained by twelve equal regular pentagons.



25. There are five, and only five, regular polyhedra (see pages 135, 136). They are sometimes spoken of as the *five regular solids;* they were known among the Greeks as *the Platonic bodies.*

Euclid omits the word "regular" in Defs. 25–29.

Instructions for making models of the five regular polyhedra will be found in *Mensuration*, by Briggs and Edmondson, pages 55–57.

26. A *cube* is a regular hexahedron because it has 6 faces.

C. A *parallelepiped* is a hexahedron whose opposite faces are parallel.

A parallelepiped is also a prism whose bases are parallelograms.

A *rectangular parallelepiped* or right-angled parallelepiped is also called an *orthohedron*.

A cube is an orthohedron whose edges are equal; a cube is also a parallelepiped whose faces are squares.

Exercises.

1. How many planes can pass through (*a*) two points, (*b*) three points, (*c*) four points? Are there any exceptional cases?

2. Name the three solids of revolution.

3. How many surfaces has each of the solids of revolution?

4. How many (*a*) corners, (*b*) faces, (*c*) edges has each of the regular polyhedra?

5. What sort of figure would be generated by the revolution of a symmetrical trapezium about the perpendicular to its parallel sides at their middle points?

6. What is the least number of edges that (*a*) a pyramid, (*b*) a prism can have?

7. Define *normal, corner, dihedral angle, vertex, re-entrant angle.*

8. Explain the meaning of the words "right," "regular," "convex," as used in Solid Geometry.

9. Name the following solid figures:—

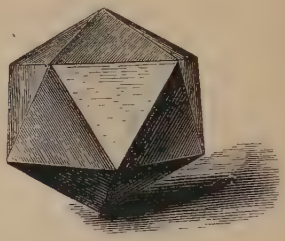
   (*a*) A polyhedron of whose faces all but one meet in a point.

   (*b*) A pyramid on a triangular base.

   (*c*) A pyramid on a circular base.

   (*d*) A polyhedron whose faces, all but two, are parallel to one straight line.

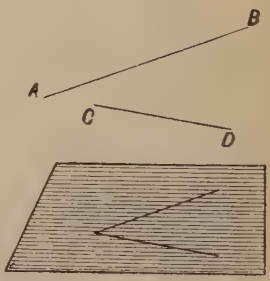
10. Define an *oblique cone* and an *oblique cylinder.*



29. A *regular icosahedron* is a solid figure contained by twenty equal equilateral triangles.

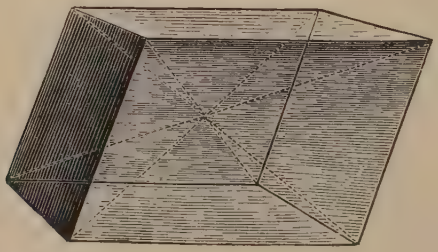
Additional Definitions.

A. A straight line is said to be parallel to a plane when the line and the plane do not meet if produced.



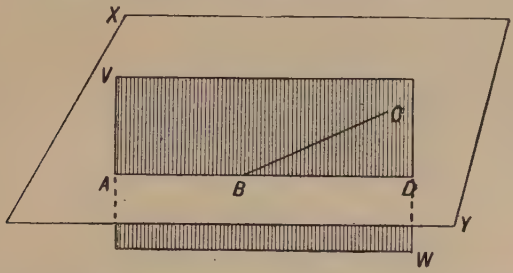
B. The angle between two straight lines which do not meet is the angle contained by two straight lines parallel to them drawn through any point.

C. A *parallelepiped* is a solid figure contained by six quadrilaterals, of which every opposite two are parallel.



***PROP. 1.**—*Theorem.*—**One part of a straight line cannot be in a plane and another part without it.**

If it be possible, let *AB*, part of the straight line *ABC*, be in the plane *XY*, and the part *BC* without *XY*.



Construction.—Since the straight line *AB* is in the plane *XY*,
∴ *AB* can be produced in the plane *XY*; *I., Post. 2*
produce *AB* to *D*.
Let another plane *VW* pass through *ABD*, and let it be turned about *AD* until it passes through *C*.

Proof.—Because *B* and *C* are two points in the plane *VW*,
∴ the straight line *BC* is in the plane *VW*;
but *ABD* is also in the plane *VW*;
∴ the two straight lines *ABC* and *ABD* are in the same plane and have a common segment *AB*, which is impossible;
∴ the straight line *ABC* cannot have one part in the plane *XY* and another part without *XY*. Q. E. D.

EXERCISES.

1. Show that, if two straight lines meet in two points, they must be in one and the same straight line.

2. A straight line cannot intersect a plane at more than one point.

3. What postulate or postulates does Euclid assume in XI. 1?

4. Prove that two straight lines in the same plane cannot have a common segment.

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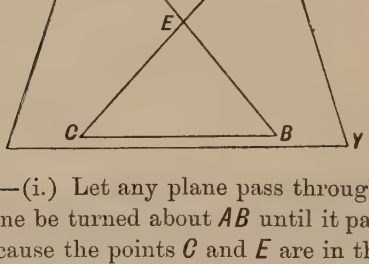
* See Note on Proposition 1, page 139.

PROP. 2.—*Theorem.*—**(i.) Two straight lines which cut one another are in one plane; and (ii.) three straight lines which meet one another in three points are in one plane.**

Let *AB* and *CD* be two straight lines which cut one another at *E*; and let *AB*, *BC*, and *CD* be three straight lines which meet one another in three points;

then (i.) *AB* and *CD* shall be in one plane,

and (ii.) *AB*, *BC*, and *CD* shall be in one plane.



Proof.—(i.) Let any plane pass through *AB*,

and let this plane be turned about *AB* until it passes through *C*;

then, because the points *C* and *E* are in this plane,

∴ the whole straight line *CED* is in this plane; *XI. 1*

∴ *AB* and *CD* are in one plane. Q. E. D.

(ii.) Again, because the points *B* and *C* are in this plane,

∴ the straight line *BC* is in this plane; *XI. 1*

∴ *AB*, *BC*, and *CD* are in one plane. Q. E. D.

Corollary.—One, and only one, plane can be drawn through two given intersecting straight lines.

EXERCISES.

1. If three straight lines not in the same plane meet one another, what other condition must they fulfil?

2. Show that the position of a plane is determined by—

(*a*) two straight lines which intersect; (*b*) the sides of a triangle;

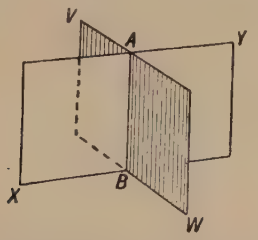
(*c*) three points not collinear; (*d*) a straight line and a point not in it;

(*e*) two parallel lines.

PROP. 3.—*Theorem.*—**If two planes cut one another, their common section is a straight line.**

Let *VW* and *XY* be two planes which cut one another in the line *AB*;

then *AB* shall be a straight line;



Proof.—Because the two points *A* and *B* are in the plane *VW*,

∴ the straight line *AB* which joins *A* and *B* lies wholly in *VW*.

I., Def. 5

Again, because the two points *A* and *B* are in the plane *XY*,

∴ the straight line *AB*, which joins *A* and *B*, lies wholly in *XY*.

I., Def. 5

∴ the straight line *AB* lies wholly in both planes;

and ∴ *AB* is the common section of the planes *VW* and *XY*;

that is, the common section of the two planes *VW* and *XY* is a straight line. Q. E. D.

EXERCISES.

1. Show that three planes may intersect (*a*) in one point, (*b*) in one straight line, or (*c*) in three parallel lines.

2. Show that, if four straight lines no three of which are in the same plane are such that each line meets two of the other lines, then the four lines all pass through one point.

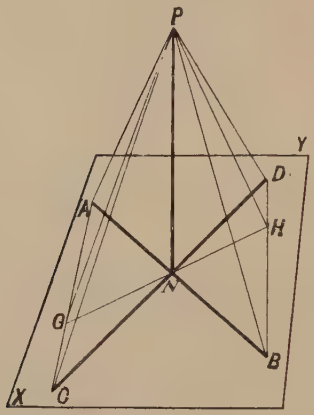
3. If two planes meet in a point, they meet in a straight line.

4. Given five points in space, what is the largest number of planes that can be drawn through them, taken three at a time?

5. Into how many parts is space divided by (*a*) one, (*b*) two, (*c*) three planes

PROP. 4.—*Theorem.*—**If a straight line be perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they are.**

Let the straight line *PN* be perpendicular to each of the straight lines *AB* and *CD* at their point of intersection *N*;
then *PN* shall also be perpendicular to the plane *XY* in which *AB* and *CD* are.



Construction. — Make *NA*, *NB*, *NC*, *ND* all equal to one another; *I. 3*
join *AC* and *BD*;
through *N* draw any straight line *GNH*, cutting *AC* and *BD* in *G* and *H* respectively;
take any point *P* in *PN*, and join *PA*, *PB*, *PC*, *PD*, *PG*, *PH*.

Proof.—(1) Because, in the two $\triangle$'s *CAN*, *DBN*,

because { *CN* = *DN*, *Constr.*
AN = *BN*, *Constr.*
and $\angle CNA = \angle DNB$, *I. 15*

$\therefore$ by I. 4, *CA* = *DB* and $\angle CAN = \angle DBN$. *I. 4*

(2) Because, in the two $\angle$'s GAN, HBN,

$$\text{because} \begin{cases} \angle GAN = \angle HBN, & \textit{Proved} \\ \angle GNA = \angle HNB, & \textit{I. 15} \\ \text{and} \quad AN = BN, \end{cases}$$

$\therefore$ by I. 26, $GA = HB$ and $GN = HN$. *I. 26*

(3) Because, in the two $\triangle$'s PAN, PBN,

$$\text{because} \begin{cases} AN = BN, \\ NP \text{ is common to both triangles,} \\ \text{and} \quad \angle PNA = \angle PNB, \text{ being right angles,} \end{cases}$$

$\therefore$ by I. 4, $PA = PB$. *I.*

Similarly, $PC = PD$.

(4) Because, in the two $\triangle$'s CAP, DBP,

$$\text{because} \begin{cases} CA = DB, \\ PA = PB, \\ \text{and} \quad PC = PD, \end{cases}$$

$\therefore$ by I. 8, $\angle CAP = \angle DBP$. *I. 8*

(5) Because, in the two $\triangle$'s GAP, HBP,

$$\text{because} \begin{cases} GA = HB, \\ AP = BP, \\ \text{and} \quad \angle GAP = \angle HBP, & \textit{Proved} \end{cases}$$

$\therefore$ by I. 4, $GP = HP$. *I. 4*

(6) Because, in the two $\triangle$'s GNP, HNP,

$$\text{because} \begin{cases} GN = HN, & \textit{Proved} \\ NP \text{ is common to both triangles,} \\ \text{and} \quad GP = HP, & \textit{Proved} \end{cases}$$

$\therefore$ by I. 8, $\angle GNP = \angle HNP$. *I. 8*

$\therefore$ PN is perpendicular to GH.

Similarly, it can be proved that PN is perpendicular to every straight line through N in the plane XY;

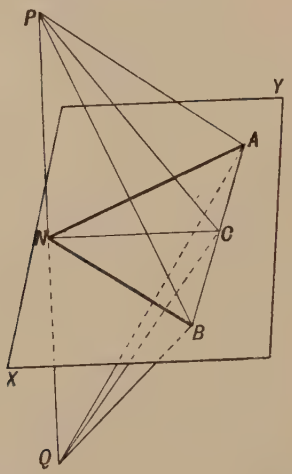
$\therefore$ PN is perpendicular to the plane XY, in which AB and CD are. *XI., Def. 3*

Q. E. D.

First Alternative Proof.

PROP. 4.—*Theorem.*—**If a straight line be perpendicular to each of two straight lines, at their point of intersection, it shall also be perpendicular to the plane in which they are.**

Let the straight line *PN* be perpendicular to each of the straight lines *NA* and *NB* at their point of intersection *N*;
then *PN* shall also be perpendicular to the plane *XY* in which *NA* and *NB* are.



Construction.—Take any point *P* in *PN*, and produce *PN* to *Q*, making *NQ* equal to *PN*;
draw any straight line *AB* in the plane *XY*, cutting *NA* and *NB*;
draw any line *NC* in the plane *XY*, meeting *AB* in *C*;
join *PA*, *PB*, *PC*, *QA*, *QB*, *QC*.

Proof.—(1) Because, in the two $\triangle$'s *PNA*, *QNA*,

because { *PN* = *QN*, *Constr.*
NA is common to both triangles,
and $\angle PNA = \angle QNA$, being right angles,

$\therefore$ by I. 4, $PA = QA$. *I. 4*
Similarly, $PB = QB$.

(2) Because, in the two $\triangle$'s PAB, QAB,
$PA = QA$, *Proved*
because AB is common to both triangles,
and $PB = QB$,
$\therefore$ by I. 8, $\angle PAB = \angle QAB$. *I. 8*

(3) Because, in the two $\triangle$'s PAC, QAC,
$PA = QA$,
because AC is common to both triangles,
and $\angle PAC = \angle QAC$, *Proved*
$\therefore$ by I. 4, $PC = QC$. *I. 4*

(4) Because, in the two $\triangle$'s PNC, QNC,
$PN = QN$,
because NC is common to both triangles,
and $PC = QC$, *Proved*
$\therefore$ by I. 8, $\angle PNC = \angle QNC$; *I. 8*
$\therefore$ each of $\angle$'s PNC, QNC is a right angle;
$\therefore$ PN is perpendicular to NC.

Similarly it can be proved that PN is perpendicular to every straight line through N in the plane XY;
$\therefore$ PN is perpendicular to the plane XY, in which NA and NB are. *XI., Def. 3*

Q. E. D.

Exercises.

1. If a straight line drawn through the centre of a circle at right angles to its plane is called the axis of the circle, prove that any point on the axis is equidistant from all points on the circumference of the circle.

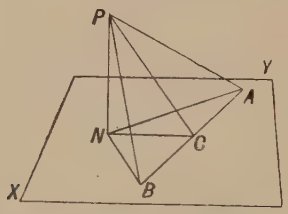
2. Prove that the locus of all points in space which are equidistant from two given points is a plane.

3. In the figure of Prop. 4, page 104, show that the angle PCN is the angle of inclination of PC to the plane XY.

Second Alternative Proof.

PROP. 4.—*Theorem.*—**If a straight line be perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they are.**

Let the straight line PN be perpendicular to each of the straight lines NA and NB at their point of intersection N;
then PN shall be perpendicular to the plane XY, in which NA and NB are.



Construction.—Take any point B in NB, and draw BA at right angles to NB in the plane XY to meet NA in A;
draw any straight line NC in the plane XY, meeting AB in C;
take any point P in NP, and join PA, PB, PC.

Proof.—Because $\angle PNA$ is a right angle,

$\therefore$ sq. on PA = sqq. on PN, NA *I. 47*
= sqq. on PN, NB, BA, $\angle NBA$ being a right angle,
= sqq. on PB, BA, $\angle PNB$ being a right angle;
$\therefore$ $\angle PBA$ is a right angle. *I. 48*

Again, because $\angle PBC$ is a right angle,

$\therefore$ sq. on PC = sqq. on PB, BC *I. 47*
= sqq. on PN, NB, BC, $\angle PNB$ being a right angle,
= sqq. on PN, NC, $\angle NBC$ being a right angle;
$\therefore$ $\angle PNC$ is a right angle; *I. 48*
that is, PN is perpendicular to NC.

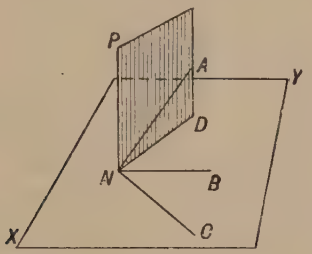
Similarly, it can be proved that PN is perpendicular to every straight line through N in the plane XY;

$\therefore$ PN is perpendicular to the plane XY, in which NA and NB are. *XI., Def. 3.* Q. E. F.

PROP. 5.—*Theorem.*—**If a straight line be perpendicular to each of three concurrent straight lines at their point of intersection, these three straight lines shall be in one and the same plane.**

Let the straight line *PN* be perpendicular to each of the three concurrent straight lines *NA*, *NB*, *NC*, at their point of intersection *N*;

then the three straight lines *NA*, *NB*, *NC* shall be in one and the same plane.



Construction.—Let *XY* be the plane which passes through *NB* and *NC*; *XI. 2*

and, if possible, let *NA* not be in the plane *XY*.

Let a plane pass through *NP* and *NA*;

and let *ND* be the common section of this plane with the plane *XY*.

Proof.—Because *PN* is perpendicular to *NB* and *NC*,

∴ *PN* is perpendicular to the plane *XY* which passes through *NB* and *NC*; *XI. 4*

∴ *PN* is perpendicular to every line which meets it in *XY*;

∴ *PN* is perpendicular to *ND*;

∴ *PND* is a right angle.

But *PNA* is a right angle; *Hyp.*

∴ $\angle PNA = \angle PND$, which are in the same plane, *Ax. 11*

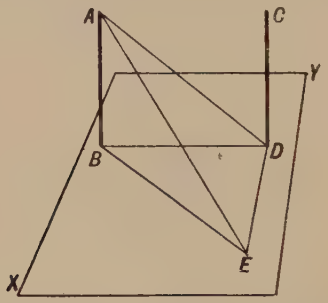
which is impossible. *Ax. 9*

∴ *NA*, *NB*, and *NC* must be in one and the same plane. Q. E. D.

PROP. 6.—*Theorem.*—**If two straight lines be perpendicular to the same plane, they shall be parallel to one another.**

Let the two straight lines AB and CD be perpendicular to the plane XY;

then AB shall be parallel to CD.



Construction.—Let AB and CD meet the plane XY in the points B and D;

join BD;

in the plane XY draw DE at right angles to BD, and make DE equal to AB;

join BE, AE, AD.

Proof.—Because, in the two $\triangle$'s ABD, EDB,

because { $AB = ED$, *Constr.*
BD is common to both triangles,
and $\angle ABD = \angle EDB$, being right angles,

$\therefore$ by I. 4, $AD = EB$. *I. 4*

Again, because, in the two $\triangle$'s EDA, ABE,

because { $ED = AB$,
$AD = BE$, *Proved*
and the base AE is common to both triangles,

$\therefore$ by I. 8, $\angle EDA = \angle ABE$; *I. 8*

but $\angle ABE$ is a right angle; *Hyp. and Constr.*

$\therefore$ $\angle EDA$ is a right angle,

and each of the ∠'s *EDB*, *EDC* is a right angle; *Constr.*
∴ the three straight lines *AD*, *BD*, and *CD* are in one plane, *XI. 5*
and *AB* is in the same plane as *AD* and *BD*; *XI. 2*
∴ *AB*, *BD*, and *DC* are in one plane.
And the two interior ∠'s *ABD*, *CDB* are two right angles;
∴ *AB* is parallel to *CD*. *I. 28*
Q. E. D.

EXERCISES ON PROPS. 4 AND 6.

1. Prove XI. 4 by supposing the line *ACB* on page 108 to be drawn so that *C* is the middle point of *AB*. (Use Prop. 5 on page 148, Bk. II., and I. 47.)

2. Equal straight lines drawn from the same point to a given plane are equally inclined to the plane.

3. At a given point in a plane only one perpendicular can be drawn to the plane.

4. The perpendicular is the least straight line that can be drawn from an external point to a plane.

5. Through a given point draw a plane perpendicular to a given plane.

6. Find the locus of points equally distant from three given points.

7. Find a point in a given straight line which shall be equidistant from any two given points in space.

8. Assuming XI. 4, what change might be made in XI., Def. 3?

9. If two straight lines in one plane be equally inclined to a second plane, then they are equally inclined to the common section of the two planes.

10. One and only one perpendicular can be drawn to a plane from a given point without the plane.

11. If two straight lines be parallel, either of them shall be parallel to every plane passing through the other line.

Is there any exceptional case?

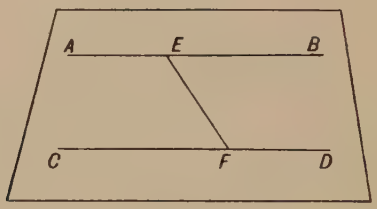
12. If a straight line be parallel to a plane, it shall also be parallel to the line of intersection of that plane with any other plane containing the straight line.

13. If two straight lines be each of them perpendicular to two parallel planes, the straight lines shall be parallel to one another.

14. If *PQ*, *QR*, and *RS* be three straight lines, such that the angles *PQR*, *QRS* are right angles, and *PQ* is perpendicular to the plane *QRS*, then *RS* shall be perpendicular to the plane *PQR*.

PROP. 7.—*Theorem.*—**If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane as the parallels.**

Let *AB* and *CD* be two parallel straight lines;
take any point *E* in *AB*, and any point *F* in *CD*;
join *EF*;
then the straight line *EF* shall be in the same plane as *AB* and *CD*.



Proof.—Because *AB* and *CD* are parallel,
∴ *AB* and *CD* are in the same plane; *I., Def. 30*
and the points *E* and *F* are in this plane,
∴ the straight line *EF*, which joins *E* and *F*, lies wholly in this plane;
that is, the straight line *EF* is in the same plane as *AB* and *CD*
Q. E. D.

Miscellaneous Exercises.

1. Three straight lines are drawn through a point mutually at right angles. Show that each is perpendicular to the plane through the other two.

2. A finite straight line revolves about another fixed straight line at right angles to it. What kind of surface does the revolving line describe, and why

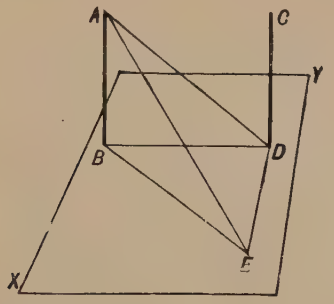
3. Show from Ex. 2 that the section of a sphere made by a plane is a circle.

4. If two straight lines be parallel to the same plane, show that they need not be parallel to one another.

5. If two planes be parallel to the same straight line, must they be paralle to one another? Give a reason for your answer.

PROP. 8.—*Theorem.*—**If two straight lines be parallel, and if one of them be perpendicular to a plane, then the other shall also be perpendicular to the same plane.**

Let *AB* and *CD* be two parallel straight lines,
and let *AB* be perpendicular to the plane *XY*;
then *CD* shall also be perpendicular to the plane *XY*.



Construction.—Let *AB* and *CD* meet the plane *XY* in the points *B* and *D*; join *BD*;
and in the plane *XY* draw *DE* at right angles to *BD*, and make *DE* equal to *AB*; join *BE*, *AE*, *AD*.

Proof.—Because, in the two △'s *ABD*, *EDB*,

because { *AB* = *ED*, *Constr*
BD is common to both triangles,
and ∠ *ABD* = ∠ *EDB*, being right angles,

∴ by I. 4, *AD* = *EB*. *I. 4*

Again, because, in the two △'s *EDA*, *ABE*,

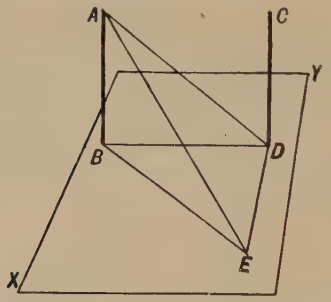
because { *ED* = *AB*,
AD = *BE*, *Proved*
and the base *AE* is common to both triangles,

∴ by I. 8, ∠ *EDA* = ∠ *ABE*; *I. 8*

but ∠ *ABE* is a right angle; *Hyp. and Constr.*

∴ ∠ *EDA* is a right angle,

that is, *ED* *is perpendicular to* *DA*.



But ED is also perpendicular to DB; *Constr.*

$\therefore$ ED is perpendicular to the plane containing DA and DB; *XI. 4*

but DC is in this plane, because DA and DB are in the same plane as the parallels AB and CD; *XI. 7*

$\therefore$ ED is perpendicular to DC, *XI., Def. 3*

that is, CD is perpendicular to DE.

Again, because AB and CD are parallel, and BD meets them,

$\therefore$ $\angle$'s ABD, CDB = two right angles; *I. 29*

but $\angle ABD$ is a right angle, *Hyp. and Constr.*

$\therefore$ $\angle CDB$ is also a right angle;

$\therefore$ CD is perpendicular to both DB and DE;

$\therefore$ CD is perpendicular to the plane containing DB and DE, that is, to the plane XY. *XI. 4*

Q. E. D.

Exercises.

1. In a given straight line find a point equidistant from two given points in space. When is this impossible?

2. In the figure of Prop. 8 prove by I. 47 that ED is perpendicular to CD, and so prove Prop. 8.

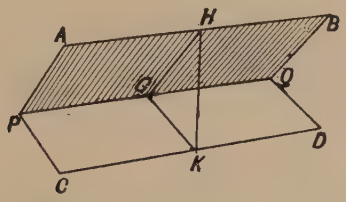
3. Prove Prop. 8, assuming Props. 6 and 12 and Playfair's axiom on parallel straight lines. (See Book I., page 70.)

4. If perpendiculars be drawn from a given point to a plane and to any straight line in the plane, show that the line joining the feet of the perpendiculars is at right angles to the straight line in the plane.

PROP. 9.—*Theorem.*—**Straight lines which are parallel to the same straight line are parallel to one another.**

Let the straight lines *AB*, *CD* be each of them parallel to *PQ*; then *AB* shall be parallel to *CD*.

CASE I. If the lines *AB*, *CD*, and *PQ* are all in one plane, the proposition has been proved in Book I. *I. 30*



CASE II. If the lines *AB*, *CD*, and *PQ* are not all in one plane:

Construction.—In *PQ* take any point *G*;

in the plane containing the parallels *AB* and *PQ*, draw *GH* at right angles to *PQ* to meet *AB* in *H*; *I. 11*

and in the plane containing the parallels *CD* and *PQ*, draw *GK* at right angles to *PQ* to meet *CD* in *K*. *I. 11*

Proof.—Because *PQ* is perpendicular to *GH* and *GK*, *Constr.*

∴ *PQ* is perpendicular to the plane containing *GH* and *GK*; *XI. 4*

but *AB* and *CD* are each of them parallel to *PQ*;

∴ *AB* and *CD* are each of them perpendicular to the same plane *GHK*; *XI. 8*

∴ *AB* is parallel to *CD*. *XI. 6*

Q. E. D.

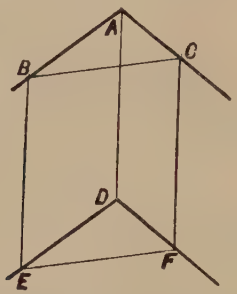
EXERCISES.

1. If the common sections of three planes are parallel, prove that the sum of the dihedral angles between the planes is equal to two right angles.

2. *PN* is perpendicular to a plane, and *NA* is drawn in the plane perpendicular to a straight line *BC* in the plane. Prove that *PA* is also perpendicular to *BC*.

PROP. 10.—*Theorem.*—**If two intersecting straight lines be respectively parallel to two other intersecting straight lines, then the first pair and the second pair shall contain equal angles.**

Let the two intersecting straight lines AB, AC be respectively parallel to the two other intersecting straight lines DE, DF; then $\angle BAC$ shall be equal to $\angle EDF$.



Construction.—Join AD:

in the plane containing the parallels AB and DE, draw BE parallel to AD, meeting AB in B and DE in E; *I. 31*

in the plane containing the parallels AC and DF, draw CF parallel to AD, meeting AC in C and DF in F; *I. 31*

join BC, EF.

Proof.—Because $ABED$ and $ADFC$ are, by construction, $\square$'s.

$\therefore$ $AB = DE$, $AC = DF$, and BE and CF each equal AD; *I. 34*

and BE and CF are also parallel to AD;

$\therefore$ $BC = EF$. *I. 33*

Again, because, in the two $\triangle$'s ABC, DEF,

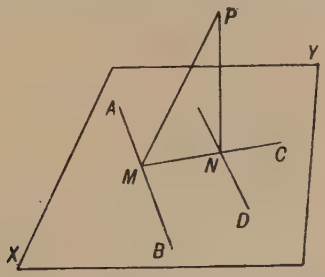
because $\begin{cases} AB = DE, & \textit{Proved} \\ AC = DF, & \textit{Proved} \\ \text{and } BC = EF, & \textit{Proved} \end{cases}$

$\therefore$ by I. 8, $\angle BAC = \angle EDF$. Q. E. D.

Corollary.—The angle of inclination of two planes is the same at all points of their common section.

PROP. 11.—*Problem.*—**To draw a straight line perpendicular to a given plane from a given point without it.**

Let *XY* be the given plane, and *P* the given point without *XY*: it is required to draw from *P* a straight line perpendicular to *XY*.



Construction.—Draw any straight line *AB* in the plane *XY*, and from *P* draw *PM* perpendicular to *AB*. *I. 12*

Then, if *PM* is perpendicular to the plane *XY*, what was required is done.

But, if not, from *M* draw *MC* in the plane *XY* at right angles to *AB*, *I. 11*

and from *P* draw *PN* perpendicular to *MC*. *I. 12*

Then *PN* shall be perpendicular to the plane *XY*.

Through *N* draw *ND* in the plane *XY* parallel to *AB*.

Proof.—Because *BM* is perpendicular to *MP* and *MN*, *Constr.*

∴ *BM* is perpendicular to the plane containing *MP* and *MN*; *XI. 4*

and *DN* is parallel to *AB*; *Constr.*

∴ *DN* is also perpendicular to the plane containing *MP* and *MN*, *XI. 8*

and *NP* is in this plane;

∴ *DN* is also perpendicular to *NP*;

∴ *PN* is perpendicular to both *NM* and *ND*;

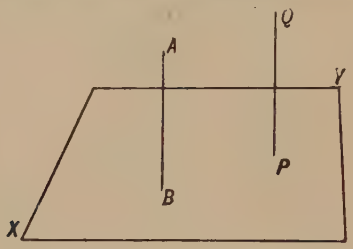
∴ *PN* is perpendicular to the plane containing *NM* and *ND*, *XI. 4*

that is, *PN* is perpendicular to the plane *XY*, and it has been drawn from the given point *P* without *XY*.

Q. E. F.

PROP. 12.—*Problem.*—**To draw a straight line at right angles to a given plane from a given point in the plane.**

Let XY be the given plane, P the point in XY;
it is required to draw from P a straight line at right angles to XY



Construction.—Take any point A outside the plane XY,
and from A draw the straight line AB perpendicular to the plane XY;
from P draw the straight line PQ parallel to AB.
Then PQ shall be at right angles to the plane XY.

Proof.—Because AB and QP are two parallel straight lines,
and one of them, AB, is perpendicular to the plane XY, *Constr.*
$\therefore$ QP is also perpendicular to the plane XY; *XI. 8*
$\therefore$ the straight line PQ has been drawn from the given point P
at right angles to the plane XY. Q. E. F.

EXERCISES.

1. If O is the orthocentre of a triangle ABC, and OP is drawn at right angles to the plane of ABC, show that PA is perpendicular to the straight line through A parallel to BC.

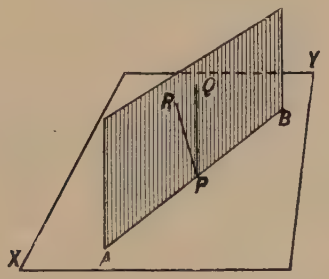
2. Straight lines are drawn from two given points A, B without a given plane, and on the same side of it, to a point P within the plane. Find when $AP + BP$ is a minimum.

3. Find the locus of the feet of perpendiculars drawn from a given point to planes which pass through another given point.

4. Find the locus of a point which moves so that it is always equidistant from two given intersecting straight lines.

PROP. 13.—*Theorem.*—**Only one perpendicular can be drawn to a given plane from a given point, whether the point be in or without the plane.**

Let XY be the given plane and P the given point;
only one perpendicular can be drawn to the plane XY from the given point P.



Case I. When the point P is in the plane XY.

If it be possible, let PQ and PR be both at right angles to the plane XY.

Let the common section of the plane passing through PQ and PR with the plane XY be the straight line AB.

Proof.—Because QP is perpendicular to the plane XY,

$\therefore$ $\angle QPA$ is a right angle.

Similarly, $\angle RPA$ is a right angle;

$\therefore$ $\angle RPA = \angle QPA$ in the same plane,

that is, the part is equal to the whole, which is absurd; *Ax. 9*

$\therefore$ PQ and PR cannot both be at right angles to the plane XY.

Case II. When the point P is without the plane XY.

If two perpendiculars could be drawn from P to the plane XY, they would be parallel to each other, *XI. 6*
which is absurd.

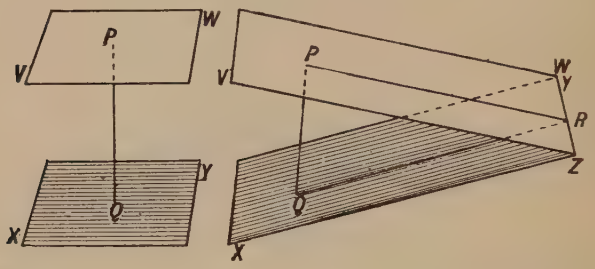
$\therefore$ only one perpendicular can be drawn to the given plane XY from the given point P, whether P be in or without XY.

Q. E. D.

PROP. 14.—*Theorem.*—**Planes to which the same straight line is perpendicular are parallel to one another.**

Let the same straight line PQ be perpendicular to each of the planes VW, XY;

then these planes shall be parallel.



Construction.—If the planes VW and XY be not parallel, they will meet, if produced.

If possible, let VW and XY, when produced, meet, and let the straight line YZ be their common section; *XI. 3*

in YZ take any point R, and join PR, QR.

Proof.—Because PQ is perpendicular to each of the planes VW, XY,

$\therefore$ the angles RPQ and RQP are two right angles,

that is, in the triangle RPQ, the two angles RPQ and RQP are together equal to two right angles, which is impossible; *I. 17*

$\therefore$ the planes VW and XY cannot be produced to meet;

$\therefore$ the planes VW and XY are parallel. Q. E. D.

EXERCISES.

1. Parallel straight lines cut the same plane at equal angles.

2. One and only one plane can be drawn through a given point parallel to a given plane.

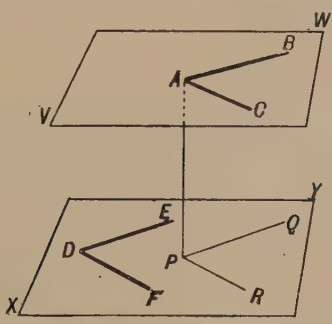
3. The projections of a straight line on parallel planes are both equal and parallel.

4. The angle of inclination of two planes is equal to the angle between any two intersecting lines perpendicular to them.

PROP. 15.—*Theorem.*—**If two intersecting straight lines in one plane be respectively parallel to two intersecting straight lines in another plane, then the two planes shall be parallel to one another.**

Let the two intersecting straight lines *AB*, *AC* in the plane *VW* be respectively parallel to the two intersecting straight lines *DE*, *DF* in the plane *XY*;

then the two planes *VW* and *XY* shall be parallel to one another.



Construction.—From *A* draw *AP* perpendicular to the plane *XY*, *XI. 11*

and let *AP* meet the plane *XY* in *P*;

through *P* in the plane *XY* draw *PQ* and *PR* respectively parallel to *DE* and *DF*. *I. 31*

Proof.—Because *AP* is perpendicular to the plane *XY*, *Constr.*

∴ each of the ∠'s *APQ*, *APR* is a right angle.

Again, because *AB* and *PQ* are each of them parallel to *DE*,

∴ *AB* is parallel to *PQ*; *XI. 9*

∴ the two ∠'s *APQ*, *PAB* are together equal to two right angles; *I. 29*

but ∠ *APQ* is a right angle, *Proved*

∴ ∠ *PAB* is also a right angle.

Similarly, it can be proved that ∠ *PAC* is a right angle.

Then, because *PA* is perpendicular to each of the lines *AB*, *AC* at their point of intersection *A*,

$\therefore$ PA is perpendicular to the plane VW, in which are AB and AC. *XI. 4*

But PA is also perpendicular to the plane XY; *Constr.*

and planes to which the same straight line is perpendicular are parallel to one another; *XI. 14*

$\therefore$ the two planes VW and XY are parallel to one another.

Q. E. D.

Miscellaneous Exercises.

1. If two straight lines in one plane be equally inclined to another plane, they will be equally inclined to the common sections of those planes.

2. Draw a straight line which shall be equally inclined to three given straight lines which meet at a point but are not all in the same plane.

3. Draw two parallel planes, one through each of two straight lines which do not intersect.

4. The straight lines which join the middle points of opposite edges of a tetrahedron meet in a point and bisect one another.

5. The projections of two equal and parallel straight lines on the same plane are themselves equal and parallel.

6. Through a given straight line draw a plane which shall be parallel to another given straight line.

7. In Prop. 15 show that the proposition is not true for parallel instead of intersecting straight lines.

8. A straight line cuts two parallel planes at the same angle.

9. ABC and $A'B'C'$ are two triangles, and the lines AA', BB', CC' are all bisected at the same point O. Prove that the triangles are in parallel planes.

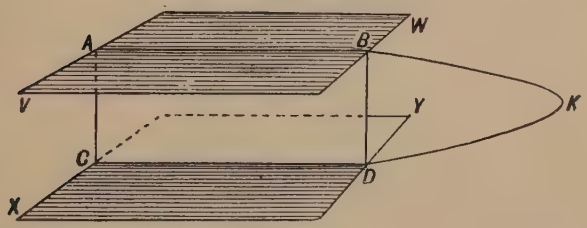
10. State and prove the converse of XI. 14.

11. Prove XI. 10 by making $AB = DE$, $AC = DF$, and joining AD, BE, CF, AC, DF.

12. PA and PB are two lines at right angles in a given plane. Let any other plane pass through PA and in it draw PC at right angles to PA. Show that the plane containing PB and PC will also contain PQ, the perpendicular from P to the given plane.

PROP. 16.—*Theorem.*—**If two parallel planes be cut by a third plane, their common sections with it shall be parallel.**

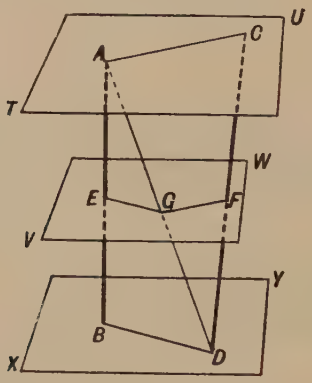
Let the two parallel planes *VW* and *XY* be cut by the third plane *ACDB*,
and let the common sections of *VW* and *XY* with *ACDB* be the straight lines *AB* and *CD*;
then *AB* shall be parallel to *CD*.



Proof.—Because *AB* and *CD* are in the same plane *ACDB*,
∴ if they are not parallel, they will meet, if produced, either towards *A* and *C* or towards *B* and *D*. *I., Def. 30*
If possible, let *AB* and *CD*, when produced, meet towards *B* and *D* in the point *K*.
Then, because the whole straight line *ABK* is in the plane *VW*, *XI. 1*
and because the whole straight line *CDK* is in the plane *XY*, *XI. 1*
∴ the planes *VW* and *XY* meet in the point *K*;
but the planes *VW* and *XY* will never meet, since they are parallel;
∴ *AB* and *CD* cannot meet when produced towards *B* and *D*.
Similarly, it may be proved that *AB* and *CD* cannot meet when produced towards *A* and *C*.
∴ *AB* is parallel to *CD*. Q. E. D.

PROP. 17.—*Theorem.*—**If two straight lines be cut by parallel planes, they shall be cut proportionately.**

Let the straight lines *AB*, *CD* be cut by the three parallel planes *TU*, *VW*, *XY* at *A*, *E*, *B* and *C*, *F*, *D*;
then shall *AE* : *EB* :: *CF* : *FD*.

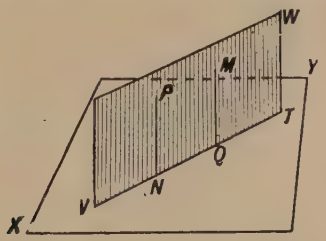


Construction.—Join *AC*, *BD*, *AD*;
and let *AD* meet the plane *VW* in *G*;
join *EG*, *FG*.

Proof.—Because the two parallel planes *VW* and *XY* are cut by the third plane *ABD*,
∴ their common sections *EG* and *BD* are parallel; *XI. 16*
and, because the two parallel planes *TU* and *VW* are cut by the third plane *ADC*,
∴ their common sections *AC* and *GF* are parallel. *XI. 16*
Again, because *EG* is parallel to *BD*, a side of △*ABD*,
∴ *AE* : *EB* :: *AG* : *GD*; *VI. 2*
and, because *GF* is parallel to *AC*, a side of △*ADC*,
∴ *AG* : *GD* :: *CF* : *FD*; *VI. 2*
∴ *AE* : *EB* :: *CF* : *FD*. *V. 11*. Q. E. D.

PROP. 18.—*Theorem.*—**If a straight line be perpendicular to a plane, then every plane which passes through the straight line shall be also perpendicular to the given plane.**

Let the straight line *PN* be perpendicular to the plane *XY*,
and let *VW* be any plane passing through *PN*;
then the plane *VW* shall be also perpendicular to the plane *XY*.



Construction.—Let *VT* be the common section of the planes *VW* and *XY*; in *VT* take any point *Q*,
and from *Q* draw *QM* in the plane *VW* at right angles to *VT*.

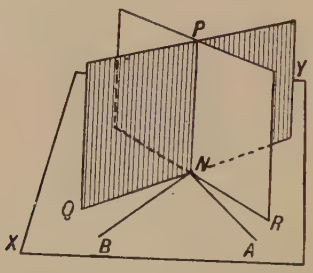
Proof.—Because *PN* is perpendicular to the plane *XY*,
$\therefore$ $\angle PNQ$ is a right angle; *XI., Def. 3*
but $\angle MQN$ is also a right angle; *Constr.*
$\therefore$ the two $\angle$'s PNQ, MQN = two right angles,
$\therefore$ *MQ* is parallel to *PN*. *I. 28*
But *PN* is perpendicular to the plane *XY*,
$\therefore$ *MQ* is also perpendicular to the plane *XY*. *XI. 8*
Similarly it can be proved that any other straight line drawn in the plane *VW* at right angles to the common section *VT* is perpendicular to the plane *XY*;
$\therefore$ the plane *VW* is perpendicular to the plane *XY*. *XI., Def. 4*
Q. E. D.

EXERCISE.

If one plane be perpendicular to a second plane, then the second plane will be perpendicular to the first plane.

PROP. 19.—*Theorem.*—**If two intersecting planes be each perpendicular to a third plane, their common section shall also be perpendicular to that plane.**

Let the two intersecting planes *PQ*, *PR* be each perpendicular to the third plane *XY*,
and let *PN* be the common section of the planes *PQ* and *PR*;
then *PN* shall be perpendicular to the plane *XY*.



Construction.—Let *QN* and *NR* be the lines of intersection of the two planes *PQ* and *PR* with the third plane *XY*;
in *XY* draw *NA* at right angles to *NQ*
and *NB* at right angles to *NR*.

Proof.—Because the plane *PQ* is perpendicular to the plane *XY*, and *NA* is drawn in the plane *XY* at right angles to their common section *QN*;

∴ *AN* is perpendicular to the plane *PQ*, *XI., Def. 4*
and *PN* is in the plane *PQ*,
∴ ∠ *PNA* is a right angle.

Similarly, because *NB* is at right angles to *NR*, it can be proved that ∠ *PNB* is a right angle,

∴ *PN* is perpendicular to the two lines *NA* and *NB* at their point of intersection;

∴ *PN* is perpendicular to the plane *XY*, in which *NA* and *NB* are. *XI. 4*

Q. E. D.

Miscellaneous Exercises.

1. Parallel planes are cut by parallel lines at the same angle.

2. If A, B, C, D be four points in space such that the lines AB, CD intersect, show that AC, BD and AD, BC will also intersect. (All the lines may be produced if necessary.)

3. Prove that, if two spheres intersect, their line of intersection is a circle.

4. AB and PQ are two straight lines, PQ being perpendicular to a plane passing through AB. Prove that all the perpendiculars to AB from different points in PQ pass through a fixed point.

5. Describe a regular tetrahedron having each of its edges equal to a given finite straight line.

6. Polygons formed by parallel planes cutting the faces of a polyhedral angle are equal.

7. Two planes are perpendicular to one another when the dihedral angle formed by them is a right angle.

8. Find the locus of points whose distances from two given planes are (*a*) equal, (*b*) in a given ratio.

9. Show that the locus of points whose distances from three intersecting planes are (*a*) equal, or (*b*) in a given ratio, is a straight line.

10. If any number of straight lines which are terminated by the same pair of parallel planes are divided in the same ratio, show that their points of section all lie in one plane.

11. Draw a plane so that its section of a given tetrahedron may be a parallelogram.

12. The straight line which is the shortest distance between two lines which do not intersect is perpendicular to both lines.

13. There is one and only one line which is perpendicular to each of two lines which neither intersect nor are parallel.

14. Bisect a given dihedral angle.

15. A and B are two points on opposite sides of a plane. Find a point P in the plane such that $PA \sim PB$ is a maximum.

16. Prove XI. 19 indirectly by XI. 13.

PROP. 20.—*Theorem.*—**If a solid angle be contained by three plane angles, any two of them shall be together greater than the third.**

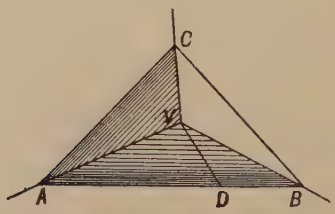
Let the solid angle at V be contained by the three plane $\angle$'s AVB, BVC, CVA;

then any two of these angles, such as AVC, CVB, shall be together greater than the third angle AVB.

CASE I. If $\angle AVB$ is equal to, or less than, either of the $\angle$'s AVC or CVB,

then it is evident that the two $\angle$'s AVC, CVB are together greater than $\angle AVB$.

CASE II. If $\angle AVB$ is greater than either of the $\angle$'s AVC or CVB.



Construction.—At the point V in the plane AVB make $\angle AVD$ equal to $\angle AVC$; *I. 23*

in VC take any point C,

and cut off VD equal to VC;

through D in the plane AVB draw any straight line ADB, meeting VA and VB in A and B; join CA, CB.

Proof.—(1) Because, in the two $\triangle$'s AVC, AVD,

because $\left\{\begin{array}{ll} VC = VD, & \textit{Constr.} \\ VA \text{ is common to both triangles,} & \\ \text{and } \angle CVA = \angle DVA, & \textit{Constr.} \end{array}\right.$

$\therefore$ by I. 4, $AC = AD$. *I. 4*

But the two sides AC, CB are together greater than AB; *I 20*

$\therefore$ CB is greater than DB.

(2) Again, because, in the two $\triangle$'s CVB, DVB,

because $\left\{\begin{array}{l} VC = VD, \\ \text{and } VB \text{ is common to both triangles,} \\ \text{but } CB \text{ is greater than } DB, \end{array}\right.$ *Proved*

$\therefore$ $\angle CVB$ is greater than $\angle DVB$. *I. 25*

But $\angle AVC = \angle AVD$, *Constr.*

$\therefore$ the two $\angle$'s AVC, CVB are together greater than the two $\angle$'s AVD, DVB, that is, than the $\angle AVB$. Q. E. D.

EXERCISES.

1. If a solid angle be contained by more than three plane angles, any one of those angles will be less than the sum of the remaining angles.

2. Prove that in a spherical triangle any two sides are together not less than the third side. (Use VI. 33.)

3. If a straight line cuts a plane, prove that the angle it makes with any straight line meeting it in that plane cannot be less than the inclination of the line to the plane nor greater than the supplement of its inclination.

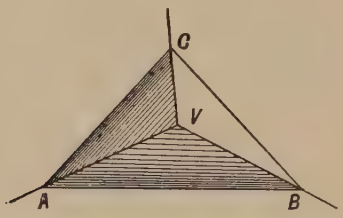
4. If a straight line makes equal angles with three straight lines meeting it in a plane, the line shall be perpendicular to the plane.

5. The faces of a tetrahedron are equal triangles. Prove that each face is an acute-angled triangle.

6. If a point O be taken within a tetrahedron $ABCD$, prove that the sum of the angles APB, BPC, CPA is greater than the sum of the angles ADB, BDC, CDA.

PROP. 21.—*Theorem.*—**All the plane angles which form a convex solid angle are together less than four right angles.**

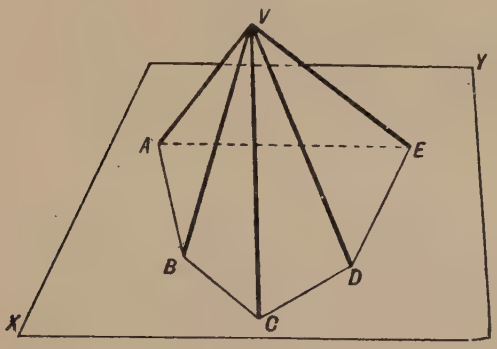
CASE I. Let the convex solid angle at V be contained by the three plane angles AVB, BVC, CVA;
the three angles AVB, BVC, CVA shall be together less than four right angles.



Construction.—In the lines VA, VB, VC take any points A, B, C; and join AB, BC, CA.

Proof.—Because the convex solid $\angle$ at A is contained by three plane angles, *XI. 20*
$\therefore$ the two $\angle$'s VAB, VAC are together greater than $\angle BAC$;
so also the two $\angle$'s VBA, VBC are together greater than $\angle ABC$,
and the two $\angle$'s VCB, VCA are together greater than $\angle BCA$;
$\therefore$ the six $\angle$'s VAB, VAC, VBA, VBC, VCB, VCA are together greater than the three $\angle$'s BAC, ABC, BCA;
but these three $\angle$'s BAC, ABC, BCA = two right angles, being the angles of the $\triangle ABC$; *I. 32*
$\therefore$ the six $\angle$'s VAB, VAC, VBA, VBC, VCB, VCA are together greater than two right angles.
But the nine angles in the three triangles VAB, VBC, VCA = six right angles, *I. 32*
and six of these nine angles are greater than two right angles;
$\therefore$ the remaining three angles, AVB, BVC, CVA, are together less than four right angles. Q. E. D.

CASE II. Let the convex solid angle at V be contained by any number of plane angles AVB, BVC, CVD, DVE, EVA; these five plane angles shall be together less than four right angles.



Construction.—Let a plane XY intersect all the planes of the five angles on the same side of the vertex V, and let AB, BC, CD, DE, EA be the common sections of the plane XY with the five planes.

Proof.—Because the convex solid $\angle$ at A is contained by three plane angles,

$\therefore$ the two angles VAB, VAE are together greater than $\angle EAB$; *XI. 20*

similarly, at each of the points B, C, D, E, the two plane angles at the bases of the triangles having the common vertex V are together greater than the third angle at the same point, which is one of the angles of the polygon $ABCDE$;

$\therefore$ all the $\angle$'s at the bases of the $\triangle$'s are greater than all the $\angle$'s of the polygon.

But all the $\angle$'s of all the $\triangle$'s are equal to twice as many right angles as there are triangles, that is, as there are sides of the polygon $ABCDE$; *I. 32*

and all the angles of the polygon, together with four right angles, are equal to twice as many right angles as there are sides of the polygon; *I. 32, Cor. 1*

∴ all the ∠'s of all the △'s = all the ∠'s of the polygon, together with four right angles.

But it has been proved that all the ∠'s at the bases of the △'s are together greater than all the ∠'s of the polygon;

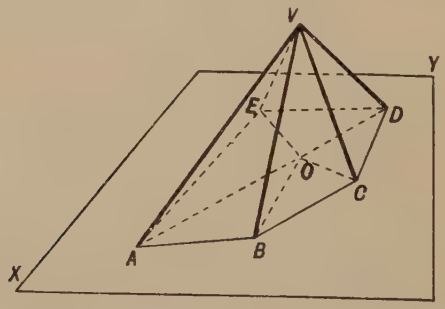
∴ the remaining ∠'s of the △'s, that is, the ∠'s at the vertex *V*, are together less than four right angles. Q. E. D.

ALTERNATIVE PROOF.

PROP. 21.—*Theorem.*—**All the plane angles which form a convex solid angle are together less than four right angles.**

Let the convex solid angle at *V* be contained by the five plane angles *AVB*, *BVC*, *CVD*, *DVE*, *EVA*;

these five plane angles shall be together less than four right angles.



Construction.—Let a plane *XY* intersect all the planes of the five angles on the same side of the vertex *V*, and let *AB*, *BC*, *CD*, *DE*, *EA* be the common sections of the plane *XY* with the five planes.

Take any point *O* within the polygon *ABCDE*, and join *OA*, *OB*, *OC*, *OD*, *OE*.

Proof.—Because the solid angle at A is contained by the three plane $\angle$'s VAE, VAB, and EAB,

$\therefore$ $\angle$'s VAE, VAB are together greater than $\angle EAB$, *XI. 20*

that is, $\angle$'s VAE, VAB are together greater than $\angle$'s OAE, OAB;

and so on for each of the corners B, C, D, E;

$\therefore$ the sum of the base angles of the triangles whose vertices are V is greater than the sum of the base angles of the triangles whose vertices are O.

But these two sets of triangles are equal in number;

$\therefore$ the sum of all the angles of the triangles with vertices V is equal to the sum of all the angles of the triangles with vertices O;

$\therefore$ the sum of all the vertical angles at V is less than the sum of all the vertical angles at O.

But the sum of the angles at O is four right angles,

I. 15, Cor. 2

$\therefore$ the sum of all the angles at the vertex V is less than four right angles. Q. E. D.

Corollary.—There cannot be more than five regular polyhedra.

Because all the plane angles which form a convex solid angle are together less than four right angles, *XI. 21*

and a solid angle is made by three or more plane angles, *XI., Def. 9*

$\therefore$ in a regular polyhedron, where all the angles are equal, each angle must be less than four-thirds of a right angle.

Again, from I. 32, Cor. 1, it is evident that the only regular figures having angles each less than four-thirds of a right angle are equilateral triangles, squares, and regular pentagons;

$\therefore$ the faces of a regular polyhedron can only be (*a*) equilateral triangles, (*b*) squares, or (*c*) pentagons.

(*a*) Each angle in an equilateral triangle is two-thirds of a right angle;

$\therefore$ only three, four, or five equilateral triangles can form a solid angle,

because six or more angles of equilateral triangles = four right angles or more;

$\therefore$ a regular polyhedron may have three, four, or five equilateral triangles to form each solid angle.

This gives three regular polyhedra, namely, the tetrahedron, the octahedron, and the icosahedron.

(b) Each angle in a square is a right angle;

$\therefore$ only three squares can form a solid angle,

because four angles of squares = four right angles.

This gives one polyhedron, the cube, in which each solid angle is formed by three squares.

(c) Each angle in a regular pentagon is six-fifths of a right angle; *I. 32, Cor. 1*

$\therefore$ only three regular pentagons can form a solid angle,

because four angles of regular pentagons are greater than four right angles.

This gives one polyhedron, the dodecahedron, in which each solid angle is formed by three regular pentagons.

$\therefore$ the five regular polyhedra are the tetrahedron, the octahedron, the icosahedron, the cube, and the dodecahedron.

Exercises.

1. If V and O be the vertices of two pyramids standing on the same convex polygon as base, and if O be within the pyramid having V as its vertex, then the sum of the plane angles at O will be less than the sum of the plane angles at V.

2. The sum of the sides of a spherical triangle is not greater than the circumference of the greatest circle which can be drawn on the sphere.

3. If from the vertex of a trihedral angle a line be drawn within it, then the sum of the angles which it makes with the edges of the trihedral angle are less than the sum, but greater than half the sum, of the plane angles which form the trihedral angle.

ADDITIONAL PROPOSITION.

If F, E, and C are the number of faces, edges, and corners respectively of any convex polyhedron, then

$$E+2=F+C.$$

Imagine the polyhedron to be built up by starting from one plane face and adding the others successively, so that the system of planes forms, at any stage, a continuous portion of the surface of the complete polyhedron.

Let the complete polyhedron have n faces, and let F_r, E_r, and C_r be the number of faces, edges, and corners respectively when r faces have been put together.

If we start with a face of a_1 sides, the number of corners and edges are equal, and we have

$$F_1=1, \quad E_1=a_1, \quad C_1=a_1; \quad \therefore \ E_1+1=F_1+C_1.$$

Now add a face of a_2 sides, having one edge in common with the first face; then the number of new edges is a_2-1, and the number of new corners is a_2-2, and we have

$$F_2=2, \quad E_2=a_1+a_2-1, \quad C_2=a_1+a_2-2;$$
$$\therefore \ E_2+1=F_2+C_2.$$

This relation holds for each addition of a face having one side only coincident with an edge of the system previously formed.

If, at any stage—say when $p-1$ faces have been put together—a face of a_p sides is added, having x adjacent sides coincident with previously existing edges, then the number of edges added is a_p-x, and the number of corners added is a_p-x-1, and we have

$$F_p=F_{p-1}+1, \quad E_p=E_{p-1}+a_p-x, \quad C_p=C_{p-1}+a_p-x-1,$$

and

$$E_{p-1}+1=F_{p-1}+C_{p-1};$$
$$\therefore \ E_p+1=F_p+C_p.$$

Hence, however many adjacent sides of the new face coincide with previously existing edges, the relation $E_r+1=F_r+C_r$ holds good, where r may have any value from 1 to $n-1$.

When the final face, which closes the polyhedron, is added no new edges or corners are added, and we have

$$F=F_{n-1}+1, \quad E=E_{n-1}, \quad C=C_{n-1},$$

and

$$E_{n-1}+1=F_{n-1}+C_{n-1}.$$

Hence

$$E+2=F+C.$$

This theorem is due to Euler. It is taken from "Mensuration," by Briggs and Edmondson, page 53.

Notes on the Propositions in Book XI.

The first six books of Euclid deal with Plane Geometry, that is, the figures are all supposed to be drawn in one plane. The Eleventh Book of Euclid serves as an introduction to Solid Geometry or the Geometry of Space.

Space is that which has length, breadth, and thickness; and these are called the three dimensions of space. Referring to the definitions in Book I., we see that a line has only one dimension, namely, length; and that a surface has two dimensions, namely, length and breadth. Books I. to VI., therefore, deal with Geometry of Two Dimensions, whereas Book XI. introduces us to Geometry of Three Dimensions.

The propositions of Book XI. may be divided into three classes, viz.:—

(i.) Lines which must be all in the same plane;
(ii.) The relations of perpendicular or parallel lines in different planes;
(iii.) The relations of two or more planes to each other.

These propositions only serve as an introduction to Solid Geometry: for, although the definitions in Book XI. deal with solids of revolution and polyhedra, yet Euclid never refers to these definitions in Book XI.

The study of Solid Geometry is rendered difficult by the manner in which the figures have to be drawn. As the lines are in different planes, the figures have to be drawn in perspective. Lines and angles, therefore, which are equal, are not necessarily represented by equal lengths or equal angles. The beginner is advised, whenever he finds a difficulty in understanding a figure, to make a simple model of all or part of the figure. In some cases, as for example in XI. 4, 6, and 7, the student will find great assistance by trying to realise the actual figure by merely holding pencils or penholders at right angles to a horizontal surface. Or, again, the student may draw separate figures for equal triangles which are situated in different planes; and he may mark equal lines and angles, so as to remember what magnitudes are equal.

Proposition 1.—This proposition hardly needs proof, and might be assumed as following at once from the definitions of a straight line and a plane surface given in Book I. The proof depends on the fact that two straight lines in the same plane cannot have a common segment. Simson proved this as a corollary to I. 11, using the method of proof adopted in I. 14.

In XI. 1 and 2 Euclid assumes that a plane can be turned round any fixed line in itself, and by this means it can be made to pass through every point in space. The fact is obvious, but it should be given as a postulate. A similar postulate is used in defining the sphere, cone, and cylinder.

Proposition 2.—The student will notice that the three lines meet in *three points*. Three straight lines which meet in one point need not be in one plane (see Prop. 5). This proposition shows how the position of any plane may be fixed (see Ex. 2, page 104).

Proposition 3.—Euclid proves this proposition by Axiom 10, that two straight lines cannot enclose a space, assuming that the planes meet in two points which are joined by straight lines. The direct proof given in the text is quite easy and depends on the definition of a plane.

Proposition 4.—This is a most important theorem and is often used. The first proof is Euclid's. It is long, but not difficult. The student may find it help him if he notices that the proof depends on I. 4, I. 8, and I. 26, and that the first triangles in each pair are (1) CAN, (2) GAN, (3) PAN, (4) CAP, (5) GAP, (6) GNP; or he may remember that he proves in (1) CA, in (2) GA, in (3) PA, in (4) $\angle CAP$, in (5) GP, and in (6) $\angle GNP$ equal to something else, and it is easy to remember these in the reverse order because he knows from the enunciation that he has to prove GNP a right angle, so that the angle GNP comes last.

The great length of Euclid's proof has suggested the advisability of finding alternative proofs. The First Alternative Proof is taken from Combette's *Cours de Géométrie Elémentaire*, and may be advantageously adopted instead of Euclid's proof. It is really only a shortened form of Euclid's proof, the first two parts of which are rendered unnecessary by drawing the figure differently. It will be seen that in the First Alternative Proof, Parts (1), (2), (3), and (4) are really the same as Parts (3), (4), (5), and (6) in Euclid's proof, and depend on I. 4 and I. 8. It may help the student to remember the order of the proofs in Prop. 4 if he notices that in (1) PA, in (2) $\angle PAB$, in (3) PC, and in (4) $\angle PNC$, is proved equal to something else, Proofs (1) and (3) depending on I. 4, and Proofs (2) and (4) depending on I. 8. The manner in which PN is produced to Q will remind the student of the important riders proved in the same way in Book I. (see page 103, Exx. 17, 18).

The Second Alternative Proof is shorter still. It is thoroughly sound, but it depends on Props. 47 and 48 instead of on earlier propositions in Book I., and the method of proof is not one used by Euclid in other propositions.

The student, therefore, is advised to use the First Alternative Proof as a regular thing. He must use Euclid's proof if that is specially asked for; and he may use the Second Alternative Proof when he is pressed for time.

A Third Alternative Proof is suggested in Ex. 1, page 113.

Propositions 6 and 8.—These are two important propositions, and are converse to each other. The proof of Prop. 6 falls into two parts:—(1) that AB and CD are in the same plane; (2) that the $\angle$'s ABD, CDB = two right angles. The proof of Prop. 8 depends on XI. 4, CD being proved to be perpendicular to both DE and DB. In both propositions the first half of the proof is the same, and in Prop. 8 "*that is, ED is perpendicular to DA*" is printed in italics because that is where the difference in the proof begins. An Alternative Proof of Prop. 8, depending on I. 47, is suggested in Ex. 3, page 116.

Students often find a difficulty in remembering the order of these propositions. They should remember that CD is proved *parallel* in Prop. 6, and *perpendicular* in Prop. 8.

Proposition 7.—This proposition is assumed as true in Book I., and it is quite evident. It is required in Prop. 8.

Proposition 9.—The enunciation of this proposition is the same as that of I. 30; but XI. 9 refers to lines in space. It is an easy rider on XI. 6 and 8.

Proposition 10.—This is an easy rider on I. 33 and I. 8. It can also be proved by making $AB = DE$ and $AC = DF$ (see Ex. 11, page 124), instead of drawing the parallels BE and CF.

Proposition 11.—This is a most important problem. The student should make himself thoroughly familiar with it. The construction, depending on I. 11 and 12, is the most important part; the proof, depending on XI. 4 and 8, is easy.

Proposition 12.—It is curious that I. 11 and 12 resemble XI. 11 and 12; but the position of the given point is *in* the line in I. 11, and *without* the plane in XI. 11, and *vice versa* in I. 12 and XI. 12. The proof of XI. 12 is evident from XI. 8.

Proposition 13.—On this proposition the theory of Projection depends; for it follows from this proposition that the projection of a point is a point, of a line is a line, and of a straight line is a straight line.

Proposition 14.—The method of proof used in this proposition recalls the method used in I. 27.

Proposition 15.—This proposition depends on proving that AP is perpendicular to the plane XY by XI. 4, and is therefore perpendicular to both planes, and so, by XI. 14, the planes are parallel. The enunciation of this proposition may be given in several different forms; but the student will easily recognise the proposition.

Proposition 16.—The student should compare the proof of this proposition with the proofs of I. 27 and of XI. 14. They are essentially the same for all three propositions.

Proposition 17.—This is an important theorem. Assuming XI. 16, it is easily proved by VI. 2.

Proposition 18.—Before learning this proposition the student should make himself quite familiar with Def. 4 in Book XI., about perpendicular planes. XI. 18 is really a rider on XI. 8.

Proposition 19.—The direct proof of this proposition is easy by means of XI. 4. The student will also find an easy indirect proof depending on XI. 13, and given by Euclid. A direct proof of a proposition is generally to be preferred; so Euclid's proof is omitted.

Propositions 20 and 21.—These important propositions deal with solid angles, and may be looked upon as an introduction to the study of solid figures. The enunciation of XI. 20 recalls the enunciation of I. 20, and the proof also of XI. 20 uses Prop. I. 20. The proof of Prop. 20 falls into two

parts. In (1) we prove, by I. 4 and I. 20, that CB is greater than DB; and in (2) $\angle CVB$ is proved greater than $\angle DVB$. Euclid's proof of Prop. 21 is long, because it involves two cases, first for a trihedral angle, and then for a polyhedral angle, and both cases depend on XI. 20 and I. 32. The Alternative Proof may be adopted by the student, because it is applicable to every kind of convex solid angle. It also depends on XI. 20. Prop. 21 is not true unless the solid angle is convex, that is, has no re-entrant angle.

If the faces of a trihedral angle at V be cut by a sphere, centre V, a spherical triangle will be formed, and, by Euclid VI. 33, the sides of this triangle will measure the angles at V. Hence we prove Ex. 2, page 131, and Ex. 2, page 136.

Summary of the Results arrived at in Euclid, Book XI.

Problems.

There are two Problems in Book XI.:—

(1) To draw a perpendicular to a given plane from a point without it. (Prop. 11.)

(2) To draw a perpendicular to a given plane from a point in the plane. (Prop. 12.)

Theorems.

The Theorems in Book XI. may be arranged in three sets, treating of—

I. Lines which must be in the same Plane;

II. The Relations of Perpendicular and Parallel Lines in different Planes;

III. The Relations of two or more Planes to each other.

I. *Lines which must be in the same Plane.*

(1) One straight line. (Prop. 1.)

(2) Two straight lines which intersect. (Prop. 2.)

(3) Three straight lines which meet in pairs. (Prop. 2.)

(4) Two parallel lines and any line cutting them. (Prop. 7.)

II. *Perpendicular and Parallel Lines in different Planes.*

(5) A line is perpendicular to a plane when—

(*a*) It is perpendicular to two lines at their point of intersection. (Prop. 4.)

(*b*) It is parallel to a line perpendicular to the plane. (Prop. 8.)

(6) Three lines which have the same perpendicular are in one plane. Prop. 5.)

(7) Lines are parallel when—
  (*a*) They are perpendicular to the same plane. (Prop. 6.)
  (*b*) They are parallel to the same straight line (Prop. 9.)
(8) Parallel lines which intersect contain equal angles. (Prop. 10.)
(9) Only one perpendicular can be drawn to a plane from one point. (Prop. 13.)

III. *The Relations of two or more Planes.*
(10) Planes are parallel when—
  (*a*) They have the same perpendicular. (Prop. 14.)
  (*b*) Intersecting lines in each are parallel. (Prop. 15.)
(11) If planes are parallel—
  (*a*) Their common sections with a third plane are parallel. (Prop. 17.)
  (*b*) Two straight lines are cut proportionally. (Prop. 17.)
(12) A plane is perpendicular to a plane if it passes through a perpendicular line. (Prop. 18.)
(13) If two planes are perpendicular to a plane, their common section is also perpendicular to the plane.
(14) If three or more planes meet at a point—
  (*a*) With a trihedral angle any two of the plane angles are together greater than the third. (Prop. 20.)
  (*b*) All the plane angles at the point are together less than four right angles. (Prop. 21.)

Miscellaneous Exercises on Book XI.

1. If the vertices of a triangle be equidistant from a point without their plane, the foot of the perpendicular from that point to the plane is the centre of the circle circumscribing the triangle.

2. If two straight lines are parallel, they are cut proportionally by three planes which have a common line of intersection.

3. Construct a trihedral angle equal to a given trihedral angle.

4. Sections of a pyramid by planes parallel to the base are similar to the base.

5. If two straight lines in one plane be equally inclined to a second plane, they will also be equally inclined to the common section of the two planes.

6. From a given point P two perpendiculars PA, PB are drawn one to each of two intersecting planes. Prove that the common section of the planes is also perpendicular to the plane containing PA, PB.

7. Prove the following properties of a tetrahedron :—

(*a*) Every section made by a plane parallel to two opposite edges is a parallelogram.

(*b*) The three straight lines which join the middle points of pairs of opposite edges meet and bisect each other.

(*c*) The four straight lines which join the vertices of a tetrahedron to the centroids of the opposite faces pass through one point, and this point divides the lines in the ratio 1 : 3.

8. Draw a straight line which shall be perpendicular to each of two straight lines which are in different planes, and prove that this perpendicular is the shortest distance between the two lines.

9. Two straight lines neither intersect nor are parallel. In what planes will their projections be parallel?

10. Show how to cut a cube by a plane so that the lines of section may form a regular hexagon.

11. The diagonals of a parallelepiped are concurrent and bisect each other.

12. The areas of sections of a pyramid made by planes parallel to the base are to one another in the duplicate ratio of their altitudes.

13. If a prism be cut by parallel planes, the sections are polygons which are equal in all respects.

14. Any section of a parallelepiped by a plane which cuts any two pairs of opposite faces is a parallelogram.

15. The four diagonals of a parallelepiped bisect each other.

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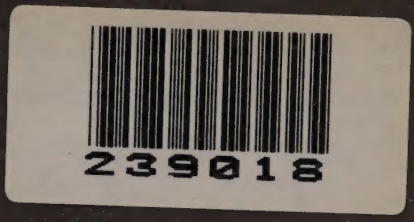
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